

UNIVERSALITY OF SAMPLE COVARIANCE MATRICES: CLT OF THE SMOOTHED EMPIRICAL SPECTRAL DISTRIBUTION

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ABSTRACT. A central limit theorem (CLT) for the smoothed empirical spectral distribution of sample covariance matrices is established. Moreover, the CLTs for the smoothed quantiles of Marcenko and Pastur's law have been also developed.

1. INTRODUCTION

The sample covariance matrix is defined by

$$\mathbf{A}_n = \frac{1}{n} \mathbf{X}_n \mathbf{X}_n^T, \text{ where } \mathbf{X}_n = (X_{ij})_{p \times n}$$

with $\{X_{ij}\}$, $i, j = \dots$, being a double array of independent and identically distributed (i.i.d.) real random variables (r.v.'s) with $\mathbb{E}X_{11} = 0$ and $\mathbb{E}X_{11}^2 = 1$. In the large dimensional random matrix theory, the sample covariance matrix is a prominent model. One reason is that its eigenvalues are not only interesting in its own right, but also play important roles in many other areas of mathematics and engineering, such as combinatorics [16], mathematical physics [18], probability [1], statistics [14] and wireless communications [19]. Its study dates back to the work of Wishart [25], who considered the case where all X_{ij} are Gaussian r.v.'s. In this particular model, the joint distribution of the eigenvalues of \mathbf{A}_n can be explicitly computed (as a special case of the Laguerre orthogonal ensemble). One can use this explicit formula to directly obtain the law of local eigenvalues in large dimensions, such as the distribution of the largest one [12, 14], the smallest one [7], and the bulk ones [23]. Also it is widely conjectured that these limiting behaviors hold for a much larger class of sample covariance matrices. For recent progress in this direction, we refer to [9]. By comparison, the universality of the Wigner matrices and β -ensembles is well developed, see [24] and [8].

However, in order to capture the whole picture of the eigenvalues of sample covariance matrices, it is necessary to study the behavior of all eigenvalues. A good candidate for

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this purpose is the empirical spectral distribution (ESD) defined by

$$F^{\mathbf{A}_n}(x) = \frac{1}{p} \sum_{k=1}^p I(\lambda_k \leq x),$$

where $\lambda_k, k = 1, \dots, p$ are the eigenvalues of \mathbf{A}_n . It is equivalent to consider

$$\mathbf{B}_n = \frac{1}{n} \mathbf{X}_n^T \mathbf{X}_n,$$

because the eigenvalues of \mathbf{A}_n and \mathbf{B}_n differ by $|n - p|$ zero eigenvalues. The almost sure convergence of $F^{\mathbf{A}_n}$ to the famous Marcenko-Pastur law (MP law) is fully understood under the 2nd moment condition of X_{11} when the dimension p is of the same order as the sample size n . There has been a vast literature on this topic. One can refer to the pioneer work [17] and the recent book [4].

After establishing the strong law of large numbers (SLLN), one may wish to prove the central limit theorem (CLT). However, as far as we know, even for the Wishart ensemble, there is no CLT available in the literature about $F^{\mathbf{A}_n}(\cdot)$ due to the shortage of powerful tools. Hence it is also impossible to make inference based on the individual eigenvalue of the sample covariance matrix when one only has finite moment conditions. These difficulties push one to seek other possible ways to make statistical inference.

Motivated by the “smoothing” ideas, Jing, Pan, Shao and Zhou [11] propose the following kernel estimators of the distribution function of the MP law,

$$(1.1) \quad F_n(x) = \int_{-\infty}^x f_n(y) dy,$$

where

$$(1.2) \quad f_n(x) = \frac{1}{ph} \sum_{i=1}^p K\left(\frac{x - \lambda_i}{h}\right) = \frac{1}{h} \int K\left(\frac{x - y}{h}\right) dF^{\mathbf{A}_n}(y),$$

$K(\cdot)$ is a smooth function and h is the bandwidth tending to zero as $n \rightarrow \infty$. Intuitively, $F_n(\cdot)$ depicts the global picture of all eigenvalues and should have no much difference from $F^{\mathbf{A}_n}(\cdot)$. It was proved in [11] that $F_n(\cdot)$ almost surely converges to the MP law under some regularity conditions.

The main aim of this paper is to establish the CLTs for $F_n(x)$, the smoothed version of the empirical spectral distribution $F^{\mathbf{A}_n}$, and $f_n(x)$. Moreover we develop CLTs for the α -th quantile of $F_n(\cdot)$, which is a smoothed version of the $[p\alpha]$ -th largest eigenvalue of \mathbf{A}_n .

2. MAIN RESULTS

We first introduce some necessary notation, assumptions and some basic facts about the MP law.

In this paper, we suppose that the ratio of the dimension and sample size $c_n = p/n$ tends to a positive constant c as $n \rightarrow \infty$. Then $F^{\mathbf{A}_n}(\cdot)$ tends to the so-called Marcenko

and Pastur law with the density function

$$f_c(x) = \begin{cases} (2\pi cx)^{-1} \sqrt{(b-x)(x-a)} & a \leq x \leq b. \\ 0 & \text{otherwise.} \end{cases}$$

It has point mass $1 - c^{-1}$ at the origin if $c > 1$, where $a = (1 - \sqrt{c})^2$ and $b = (1 + \sqrt{c})^2$ (see [4]). The distribution function of the MP law is denoted by $\mathbb{F}_c(\cdot)$. The Stieltjes transform of the MP law is

$$(2.1) \quad m(z) = \frac{1 - c - z + \sqrt{(z - 1 - c)^2 - 4c}}{2cz},$$

which satisfies the equation

$$(2.2) \quad m(z) = \frac{1}{1 - c - czm(z) - z}.$$

Here the Stieltjes transform $m_F(\cdot)$ for any probability distribution function $F(\cdot)$ is given by

$$(2.3) \quad m_F(z) = \int \frac{1}{x - z} dF(x), \quad z \in \mathcal{C}^+.$$

The relationship between the Stieltjes transform of the limit of $F^{\mathbf{B}_n}(\cdot)$ and $m(\cdot)$ is given by

$$(2.4) \quad \underline{m}(z) = -\frac{1 - c}{z} + cm(z).$$

which gives the equation satisfied by $\underline{m}(\cdot)$

$$(2.5) \quad z = -\frac{1}{\underline{m}(z)} + \frac{c}{1 + \underline{m}(z)}.$$

For the kernel function $K(\cdot)$ we assume that

$$(2.6) \quad \lim_{|x| \rightarrow \infty} |xK(x)| = \lim_{|x| \rightarrow \infty} |xK'(x)| = 0,$$

$$(2.7) \quad \int K(x) dx = 1, \quad \int |xK'(x)| dx < \infty, \quad \int |K''(x)| dx < \infty.$$

and

$$(2.8) \quad \int xK(x) dx = 0, \quad \int x^2 |K(x)| dx < \infty.$$

Let $z = u + iv$, where $u \in \mathbb{R}$ and v is in a bounded interval, say $[-v_0, v_0]$ with $v_0 > 0$. Suppose that

$$(2.9) \quad \int_{-\infty}^{+\infty} |K^{(j)}(z)| du < \infty, \quad j = 0, 1, 2,$$

uniformly in $v \in [-v_0, v_0]$, where $K^{(j)}(z)$ denotes the j -th derivative of $K(z)$. Also suppose that

$$(2.10) \quad \lim_{|x| \rightarrow \infty} |xK(x + iv_0)| = \lim_{|x| \rightarrow \infty} |xK'(x + iv_0)| = 0.$$

Our first result is the CLT for $(F_n(x) - \mathbb{F}_{c_n}(x))$.

Theorem 1. *Suppose that*

- 1) $h = h(n)$ is a sequence of positive constants satisfying

$$\lim_{n \rightarrow \infty} \frac{nh^2}{\sqrt{\ln h^{-1}}} \rightarrow 0, \quad \lim_{n \rightarrow \infty} \frac{1}{nh^2} \rightarrow 0,$$

- 2) $K(x)$ satisfies (2.6)-(2.10) and is analytic on open interval including

$$\left[\frac{a-b}{h}, \frac{b-a}{h}\right];$$

- 3) X_{ij} are i.i.d. with $\mathbb{E}X_{11} = 0$, $\text{Var}(X_{11}) = 1$, $\mathbb{E}X_{11}^4 = 3$ and $\mathbb{E}X_{11}^{32} < \infty$, $c_n \rightarrow c \in (0, 1)$;

Then, as $n \rightarrow \infty$, for any fixed positive integer d and different points x_1, \dots, x_d in (a, b) , the joint limiting distribution of

$$(2.11) \quad \frac{\sqrt{2\pi n}}{\sqrt{\ln n}} \left(F_n(x_j) - \mathbb{F}_{c_n}(x_j) \right), \quad j = 1, \dots, d$$

is multivariate normal with mean zero and covariance matrix I , the $d \times d$ identity matrix.

Remark 1. The convergence rate $n/\sqrt{\ln n}$ is consistent with the conjectured convergence rate $n/\sqrt{\ln n}$ of the ESD of sample covariance matrices to the MP law.

Remark 2. It is easy to check that the Gaussian kernel $(2\pi)^{-1/2}e^{-x^2/2}$ satisfies all conditions specified in Theorem 1.

Based on Theorem 1 we may further develop the smoothed quantile estimators of the MP law. For $0 < \alpha < 1$, define the α -quantile of the MP law by

$$(2.12) \quad x_\alpha = \inf\{x, \mathbb{F}_{c_n}(x) \geq \alpha\}$$

and its estimator by

$$(2.13) \quad x_{n,\alpha} = \inf\{x, F_n(x) \geq \alpha\}.$$

Theorem 2. *Under the assumptions of Theorem 1,*

$$\frac{n}{\sqrt{\ln n}}(x_{n,\alpha} - x_\alpha) \rightarrow N\left(0, \frac{1}{2\pi^2 f_c^2(x_\alpha)}\right), \quad x_\alpha \in (a, b).$$

The next theorem is the CLT for $f_n(x)$.

Theorem 3. *Suppose that*

- 1) $h = h(n)$ is a sequence of positive constants satisfying

$$(2.14) \quad \lim_{n \rightarrow \infty} \frac{\ln h^{-1}}{nh^2} \rightarrow 0, \quad \lim_{n \rightarrow \infty} nh^3 = 0;$$

- 2) $K(x)$ satisfies (2.6)-(2.10) and is analytic on open interval including

$$\left[\frac{a-b}{h}, \frac{b-a}{h}\right];$$

- 3) X_{ij} are i.i.d. with $\mathbb{E}X_{11} = 0$, $\text{Var}(X_{11}) = 1$, $\mathbb{E}X_{11}^4 = 3$ and $\mathbb{E}X_{11}^{32} < \infty$, $c_n \rightarrow c \in (0, 1)$;

Then, as $n \rightarrow \infty$, for any fixed positive integer d and different points x_1, \dots, x_d in (a, b) , the joint limiting distribution of

$$(2.15) \quad nh \left(f_n(x_j) - f_{c_n}(x_j) \right), \quad j = 1, \dots, d$$

is multivariate normal with mean zero and covariance matrix $\sigma^2 I$, where

$$\sigma^2 = -\frac{1}{2\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1)K'(u_2) \ln(u_1 - u_2)^2 du_1 du_2.$$

Note that the Gaussian kernel $(2\pi)^{-1/2} e^{-x^2/2}$ also satisfies all conditions specified in Theorem 3. Theorem 3 is actually a corollary of the following theorem.

Theorem 4. When the condition $\lim_{n \rightarrow \infty} nh^3 = 0$ in Theorem 3 is replaced by

$$\lim_{n \rightarrow \infty} h = 0$$

while the remaining conditions are unchanged, Theorem 3 holds as well if the random variables (2.15) are replaced by

$$nh \left[f_n(x_j) - \frac{1}{h} \int_a^b K\left(\frac{x_j - y}{h}\right) d\mathbb{F}_{c_n}(y) \right], \quad x_j \in (a, b), \quad j = 1, \dots, d$$

The paper is organized as follows. Theorem 4 is proved in Section 3, and some calculations involved in the proof are deferred to Appendix 2. In Section 4, we establish the optimal orders for $\mathbb{E}(\Gamma(z))^2$ and $\mathbb{E}(\Gamma(z))^3$ where $\Gamma(z) = n^{-1} \text{tr} \mathbf{A}^{-1}(z) - n^{-1} \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)$ with $\mathbf{A}^{-1}(z) = (\mathbf{A}_n - z\mathbf{I})^{-1}$, $z = u + iv$ and $v \geq M/\sqrt{n}$ for some constant M . It is the most difficult and important result of this paper. In Section 5, we derive the limit of $(2\pi i)^{-1} \oint K((x - z)/h) n(\mathbb{E} m_n(z) - m_n^0(z)) dz$, which is essential to Theorem 1. The proof of Theorems 3 and 1 is completed in Section 6. Section 7 handles Theorem 2. Some technical lemmas are given in Appendix 1.

Before concluding this section, let us say a few words about the proof of Theorem 1. Key breakthroughs are to establish optimal orders for $\mathbb{E}(\Gamma(z))^2$ and $\mathbb{E}(\Gamma(z))^3$ with $z = u + iv$ and $v \geq M/\sqrt{n}$ for some constant M . This turns out to be quite challenging when v is of the order $n^{-1/2}$. The best order obtained so far is $\mathbb{E}|\Gamma(z)|^2 \leq M(nv)^{-2}|z + c - 1 + 2zcm(z)|^{-2}$, or $M/(n^2v^3)$ (see Proposition 6.1 in [10]). Roughly speaking we establish

$$\frac{1}{|z + c - 1 + zcm(z) + zc\mathbb{E}m_n(z)|} |\mathbb{E}(\Gamma(z))^2| \leq \frac{M}{n^2v^2|z + c - 1 + 2zcm(z)|}$$

and

$$|\mathbb{E}(\Gamma(z))^3| \leq M/(n^2v^2).$$

To this end, we develop a precise order of $n^{-1} \mathbb{E} \text{tr} \mathbf{A}^{-2}(z)$, which is of $v^{-1/2}$. Also, some sharp bound for $n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-1}(z)$ is established (the definition of $\underline{\mathbf{A}}_k^{-1}(z)$ is given

right before section 3.1). Indeed, these results also imply that the convergence rate of $\mathbb{E}m_n(z)$ to $m(z)$ is $M/(nv)$ (see Proposition 1), which further implies

$$|\mathbb{E}(\Gamma(z))^2| \leq M(nv)^{-2}.$$

We expect that this inequality could be used to solve the universality problem of the largest eigenvalue of sample covariance matrices as Johansson does in [13].

3. FINITE DIMENSIONAL CONVERGENCE OF THE PROCESSES

Throughout the paper, to save notation, M may stand for different constants on different occasions. This section deals with Theorem 4.

Following the truncation steps in [3] we may truncate and re-normalize the random variables so that

$$(3.1) \quad |X_{ij}| \leq \tau_n n^{1/2}, \quad \mathbb{E}X_{ij} = 0, \quad \mathbb{E}X_{ij}^2 = 1,$$

where $\tau_n n^{1/3} \rightarrow \infty$ and $\tau_n \rightarrow 0$. Based on this one may then verify that

$$(3.2) \quad \mathbb{E}X_{11}^4 = 3 + O\left(\frac{1}{n}\right).$$

Let $m_n^0(z)$ denote the one obtained from $m(z)$ with c replaced by c_n . For $x \in [a, b]$, by Cauchy's formula, with probability one for sufficiently large n ,

$$(3.3) \quad nh \left(\left(f_n(x) - \frac{1}{h} \int K\left(\frac{x-y}{h}\right) d\mathbb{F}_{c_n}(y) \right) \right) = -\frac{1}{2\pi i} \oint_{\mathcal{C}_1} K\left(\frac{x-z}{h}\right) X_n(z) dz,$$

where $X_n(z) = \text{tr}(\mathbf{A}_n - z\mathbf{I})^{-1} - nm_n^0(z)$ and the contour \mathcal{C}_1 is the union of four segments $\gamma_j, j = 1, 2, 3, 4$. Here

$$\gamma_1 = u - iv_0 h, u \in [a_l, a_r], \quad \gamma_2 = u + iv_0 h, u \in [a_l, a_r],$$

$$\gamma_3 = a_l + iv, v \in [-v_0 h, v_0 h], \quad \gamma_4 = a_r + iv, v \in [-v_0 h, v_0 h],$$

where a_l is any positive value smaller than a , a_r any value larger than b , and v_0 is a constant specified in (2.9).

For the sake of simplicity, write $\mathbf{A} = \mathbf{A}_n$. We now introduce some notation and present some basic facts frequently used in this paper.. Define $\mathbf{A}(z) = \mathbf{A} - z\mathbf{I}$, $\mathbf{A}_k(z) = \mathbf{A}(z) - \mathbf{s}_k \mathbf{s}_k^T$ with $n^{1/2} \mathbf{s}_k$ being the k th column of \mathbf{X}_n . Let $\mathbb{E}_k = \mathbb{E}(\cdot | \mathbf{s}_1, \dots, \mathbf{s}_k)$ and \mathbb{E}_0 denote the expectation. Let $v = \Im(z)$. Set

$$\beta_k(z) = \frac{1}{1 + \mathbf{s}_k^T \mathbf{A}_k^{-1}(z) \mathbf{s}_k}, \quad \eta_k(z) = \mathbf{s}_k^T \mathbf{A}_k^{-1}(z) \mathbf{s}_k - \frac{1}{n} \text{tr} \mathbf{A}_k^{-1}(z),$$

$$b_1(z) = \frac{1}{1 + n^{-1} \mathbb{E} \text{tr} \mathbf{A}_1^{-1}(z)}, \quad \beta_k^{\text{tr}}(z) = \frac{1}{1 + n^{-1} \text{tr} \mathbf{A}_k^{-1}(z)}.$$

$$\Gamma_k = n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) - n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z), \quad \Gamma_k^{(2)} = n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) - n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-2}(z)$$

and

$$\eta_k^{(2)}(z) = \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k - n^{-1} \text{tr} \mathbf{A}_k^{-2}(z).$$

We frequently use the following equalities:

$$(3.4) \quad \mathbf{A}^{-1}(z) - \mathbf{A}_k^{-1}(z) = -\beta_k(z) \mathbf{A}_k^{-1}(z) \mathbf{s}_k \mathbf{s}_k^T \mathbf{A}_k^{-1}(z);$$

$$(3.5) \quad \beta_k = b_1 - b_1 \beta_k \xi_k(z) = b_1 - b_1^2 \xi_k(z) + b_1^2 \beta_k \xi_k^2(z)$$

where $\xi_k(z) = \mathbf{s}_k^T \mathbf{A}_k^{-1}(z) \mathbf{s}_k - n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z)$. At this moment, we would point out that the length of the vertical lines of the contour of integral in (3.3) converges to zero. As a consequence, except $|b_1(z)|$ we can expect neither $|\beta_k(z)|$ nor $|\beta_k^{\text{tr}}(z)|$ to be bounded above by constants independent of v although they are bounded by $|z|/|v|$ (see [2]) (of course $v \neq 0$ in the cases of interest). Instead, the absolute moments of $\beta_k(z)$ and $\beta_k^{\text{tr}}(z)$ are proved to be bounded. We summarize such estimates in Lemma 8 in Appendix 1. Sometimes we deal with the terms $\beta_k^{\text{tr}}(z)$ and $\beta_k(z)$ in the following way: One may verify that

$$\Im(1 + n^{-1} \text{tr} \mathbf{A}_k^{-1}(z)) \geq v n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{A}_k^{-1}(\bar{z}),$$

which implies that

$$(3.6) \quad |\beta_k^{\text{tr}}(z) n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{A}_k^{-1}(\bar{z})| \leq M |v|^{-1}.$$

Similarly,

$$(3.7) \quad |\beta_k \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k| \leq |v|^{-1}.$$

We shall also use the simple fact that

$$(3.8) \quad \|\mathbf{A}_k^{-1}(z)\| \leq 1/|v|.$$

Throughout the paper the variable z sometimes will be dropped from their corresponding expressions when there is no confusion.

Here is the famous martingale decomposition in the random matrix theory,

$$\begin{aligned} \text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z) &= \sum_{k=1}^n (\mathbb{E}_k \text{tr} \mathbf{A}^{-1}(z) - \mathbb{E}_{k-1} \text{tr} \mathbf{A}^{-1}(z)) \\ &= \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \text{tr} [\mathbf{A}^{-1}(z) - \mathbf{A}_k^{-1}(z)] = - \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) [\beta_k(z) \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k] \\ (3.9) \quad &= - \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) [\ln \beta_k(z)]', \end{aligned}$$

where the third step uses (3.4) and the derivative in the last equality is with respect to z . We then obtain from integration by parts that

$$\begin{aligned} (3.10) \quad & \frac{1}{2\pi i} \oint K\left(\frac{x-z}{h}\right) (\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)) dz \\ &= - \frac{1}{2\pi i} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint K\left(\frac{x-z}{h}\right) [\ln \beta_k(z)]' dz \end{aligned}$$

$$(3.11) \quad = \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint K'\left(\frac{x-z}{h}\right) \ln \left(\frac{\beta_k^{\text{tr}}(z)}{\beta_k(z)} \right) dz$$

$$\begin{aligned}
&= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint K'(\frac{x-z}{h}) \ln(1 + \beta_k^{\text{tr}}(z)\eta_k(z)) dz \\
(3.12) \quad &= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint K'(\frac{x-z}{h}) (\beta_k^{\text{tr}}(z)\eta_k(z) + e_k(z)) dz
\end{aligned}$$

where the complex logarithm functions can be selected as their respective principal value branches by Cauchy's theorem and

$$e_k(z) = \ln(1 + \beta_k^{\text{tr}}(z)\eta_k(z)) - \beta_k^{\text{tr}}(z)\eta_k(z).$$

Below, consider $z \in \gamma_2$, the top horizontal line of the contour, unless it is further specified. We remind readers that $v = v_0 h$ on γ_2 . The next aim is to prove that

$$(3.13) \quad \frac{1}{h} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \int K'(\frac{x-z}{h}) e_k(z) dz \xrightarrow{i.p.} 0,$$

where *i.p.* means “in probability”. By Lemma 7, we have for $m = 2, 4, 6, 8$

$$(3.14) \quad \mathbb{E}(|\eta_k(z)|^m |\mathbf{A}_k^{-1}(z)|) \leq M n^{-m/2} [n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{A}_k^{-1}(\bar{z})]^{m/2}.$$

This, together with Lemma 8 in Appendix 1 and (3.6), gives

$$(3.15) \quad \mathbb{E}|\beta_k^{\text{tr}}(z)\eta_k(z)|^8 = \mathbb{E}(|\beta_k^{\text{tr}}(z)|^8 \mathbb{E}(|\eta_k(z)|^8 | \mathbf{A}_k^{-1}(z))) \leq M(nv)^{-4}.$$

Via (2.9), (3.15) and the inequality

$$(3.16) \quad |\ln(1+x) - x| \leq M|x|^2, \text{ for } |x| \leq 1/2,$$

we obtain

$$\begin{aligned}
&h^{-2} \mathbb{E} \left| \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \int K'(\frac{x-z}{h}) e_k(z) I(|\beta_k^{\text{tr}}(z)\eta_k(z)| < 1/2) du \right|^2 \\
(3.17) \quad &\leq M h^{-2} \sum_{k=1}^n \mathbb{E} \left| \int K'(\frac{x-z}{h}) e_k(z) I(|\beta_k^{\text{tr}}(z)\eta_k(z)| < 1/2) du \right|^2 \\
&\leq M h^{-2} \sum_{k=1}^n \left[\int \int |K'(\frac{x-z_1}{h}) K'(\frac{x-z_2}{h})| \left(\mathbb{E}(|\beta_k^{\text{tr}}(z_1)\eta_k(z_1)|)^4 \right. \right. \\
&\quad \left. \left. \times \mathbb{E}(|\beta_k^{\text{tr}}(z_2)\eta_k(z_2)|)^4 \right)^{1/2} du_1 du_2 \right] \leq M/(nv^2).
\end{aligned}$$

Note that

$$\ln(1 + \beta_k^{\text{tr}}(z)\eta_k(z)) = \ln \beta_k(z) - \ln \beta_k^{\text{tr}}(z).$$

Moreover $|\beta_k(z)| \leq |z|/v$ and

$$|\beta_k(z)| \geq (1 + v^{-1} \mathbf{s}_k^T \mathbf{s}_k)^{-1} \geq (1 + v^{-1} n \tau_n)^{-1}.$$

It follows that

$$|\ln \beta_k(z)| \leq M \max(\ln v^{-1}, \ln(n/v)).$$

Likewise $|\ln \beta_k^{\text{tr}}(z)| \leq M \ln v^{-1}$. Hence

$$|\ln(1 + \beta_k^{\text{tr}}(z)\eta_k(z))| \leq M \max(\ln v^{-1}, \ln(n\tau_n/v)).$$

This, together with (3.15), ensures that

$$\frac{1}{h} \mathbb{E} \left| \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \int K' \left(\frac{x-z}{h} \right) e_k(z) I(|\beta_k^{\text{tr}}(z) \eta_k(z)| \geq 1/2) du \right| \leq M/(nv^2).$$

Thus, (3.13) is proven. Similarly, by (3.14) and Lemma 8 we have

$$\mathbb{E} \left| \sum_{k=1}^n \mathbb{E}_k \left[\frac{1}{h} \int K' \left(\frac{x-z}{h} \right) \left((\beta_k^{\text{tr}}(z) - b_1(z)) \eta_k(z) \right) dz \right] \right|^2 \leq M/(nv^2).$$

Therefore on γ_2

$$(3.18) \quad (3.10) = \frac{1}{2\pi i} \sum_{k=1}^n Y_k(x) + o_p(1),$$

where

$$Y_k(x) = b_1(z) \mathbb{E}_k \left[\frac{1}{h} \int K' \left(\frac{x-z}{h} \right) \eta_k(z) dz \right].$$

Apparently, $Y_k(x)$ is a martingale difference so that we may resort to the CLT for martingales (see Theorem 35.12 in [5]). Here we consider only one point x . But from the late proof, one can see that we actually prove the finite dimensional convergence.

As in (3.17), by (2.9) and (3.15) we have

$$\sum_{k=1}^n \mathbb{E} |Y_k(x)|^4 \leq M/(nv^2).$$

which ensures that the Lyapunov condition in the CLT is satisfied.

Thus, it is sufficient to investigate the limit of the following covariance function

$$(3.19) \quad \begin{aligned} & -\frac{1}{4\pi^2} \sum_{k=1}^n \mathbb{E}_{k-1} [Y_k(x_1) Y_k(x_2)] \\ &= -\frac{1}{4h^2\pi^2} \int \int K' \left(\frac{x_1 - z_1}{h} \right) K' \left(\frac{x_2 - z_2}{h} \right) \mathcal{C}_{n1}(z_1, z_2) dz_1 dz_2, \end{aligned}$$

where

$$\mathcal{C}_{n1}(z_1, z_2) = b_1(z_1) b_1(z_2) \sum_{k=1}^n \mathbb{E}_{k-1} [\mathbb{E}_k(\eta_k(z_1)) \mathbb{E}_k(\eta_k(z_2))].$$

Note that for any non-random matrices \mathbf{B} and \mathbf{C}

$$(3.20) \quad \begin{aligned} & \mathbb{E}(\mathbf{s}_1^T \mathbf{C} \mathbf{s}_1 - \text{tr} \mathbf{C})(\mathbf{s}_1^T \mathbf{B} \mathbf{s}_1 - \text{tr} \mathbf{B}) \\ &= n^{-2} (\mathbb{E} X_{11}^4 - |\mathbb{E} X_{11}^2|^2 - 2) \sum_{i=1}^p (\mathbf{C})_{ii} (\mathbf{B})_{ii} + |\mathbb{E} X_{11}^2|^2 n^{-2} \text{tr} \mathbf{C} \mathbf{B}^T + n^{-2} \text{tr} \mathbf{C} \mathbf{B}. \end{aligned}$$

This implies that

$$\begin{aligned}
& b_1(z_1)b_1(z_2) \sum_{k=1}^n \mathbb{E}_{k-1}(\mathbb{E}_k \eta_k(z_1) \mathbb{E}_k \eta_k(z_2)) \\
(3.21) \quad &= (\mathbb{E} X_{11}^4 - 3)b_1(z_1)b_1(z_2)\mathcal{C}_{n1}^{(1)}(z_1, z_2) + 2b_1(z_1)b_1(z_2)\mathcal{C}_{n2}(z_1, z_2) \\
(3.22) \quad &= 2b_1(z_1)b_1(z_2)\mathcal{C}_{n2}(z_1, z_2) + O(1/(nv^2)),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}_{n1}^{(1)}(z_1, z_2) &= \frac{1}{n^2} \sum_{k=1}^n \sum_{j=1}^p \mathbb{E}_k(\mathbf{A}_k^{-1}(z_1))_{jj} \mathbb{E}_k(\mathbf{A}_k^{-1}(z_2))_{jj}, \\
\mathcal{C}_{n2}(z_1, z_2) &= \frac{1}{n^2} \sum_{k=1}^n \text{tr} \mathbb{E}_k(\mathbf{A}_k^{-1}(z_1)) \mathbb{E}_k(\mathbf{A}_k^{-1}(z_2)) \\
&= \frac{1}{n^2} \sum_{k=1}^n \text{tr} \mathbb{E}_k(\mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2))
\end{aligned}$$

and the last step uses (6.2) and (3.8). Here $\underline{\mathbf{A}}_k^{-1}(z)$ is defined by $\mathbf{s}_1, \dots, \mathbf{s}_{k-1}, \underline{\mathbf{s}}_{k+1}, \dots, \underline{\mathbf{s}}_n$ as $\mathbf{A}_k^{-1}(z)$ is defined by $\mathbf{s}_1, \dots, \mathbf{s}_{k-1}, \mathbf{s}_{k+1}, \dots, \mathbf{s}_n$ with $\underline{\mathbf{s}}_1, \dots, \underline{\mathbf{s}}_n$ being i.i.d. copies of \mathbf{s}_1 and independent of $\{\mathbf{s}_j, j = 1, \dots, n\}$.

3.1. The limit of $\mathcal{C}_{n2}(z_1, z_2)$. The next aim is to develop the limit of $\mathcal{C}_{n2}(z_1, z_2)$. To this end, we introduce more notation and estimates. Let

$$\begin{aligned}
\mathbf{A}_{kj}(z) &= \mathbf{A}(z) - \mathbf{s}_k \mathbf{s}_k^T - \mathbf{s}_j \mathbf{s}_j^T, \quad \beta_{kj}(z) = \frac{1}{1 + \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j}, \\
b_{12}(z) &= \frac{1}{1 + n^{-1} \mathbb{E} \text{tr} \mathbf{A}_{12}^{-1}(z)}, \quad \beta_{kj}^{tr}(z) = \frac{1}{1 + n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z)}, \\
\Gamma_{kj} &= n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z) - \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z), \quad \Gamma_{kj}^{(2)} = n^{-1} \text{tr} \mathbf{A}_{kj}^{-2}(z) - \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-2}(z)
\end{aligned}$$

and

$$\xi_{kj}(z) = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z), \quad \eta_{kj}(z) = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z).$$

Actually, they are similar to $\mathbf{A}_k(z), \beta_k(z), \dots$. Note that

$$(3.23) \quad \mathbf{A}_k^{-1}(z) - \mathbf{A}_{kj}^{-1}(z) = -\beta_{kj}(z) \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z)$$

which is similar to (3.4), and (see Lemma 2.10 of [2]) for any $p \times p$ matrix \mathbf{D}

$$(3.24) \quad |\text{tr}(\mathbf{A}_k^{-1}(z) - \mathbf{A}_{kj}^{-1}(z)) \mathbf{D}| \leq \|\mathbf{D}\| v^{-1}.$$

Also, we have

$$(3.25) \quad |\beta_{kj}| \|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z)\|^2 = |\beta_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \mathbf{s}_j| \leq v^{-1}.$$

Write

$$(3.26) \quad \beta_{kj}(z) = b_{12}(z) - \beta_{kj}(z) b_{12}(z) \xi_{kj}(z) = b_{12}(z) - b_{12}^2(z) \xi_{kj}(z) + \beta_{kj}(z) b_{12}^2(z) \xi_{kj}^2(z).$$

By Lemma 8 we have

$$(3.27) \quad \mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z) \mathbf{A}^{-1}(\bar{z}) = v^{-1} \Im(\mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z)) \leq M v^{-1},$$

which, together with (3.24), implies that

$$(3.28) \quad \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \leq M v^{-1}.$$

By Lemma 8 in Appendix 1 and (3.24) we then have

$$(3.29) \quad \mathbb{E} |n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z)|^4 \leq M.$$

By (3.24) and the fact that $b_1(z)$ is bounded, given in Lemma 8, it is straightforward to verify that $|b_1(z) - b_{12}(z)| \leq (nv^2)^{-1}$ and hence

$$(3.30) \quad |b_{12}(z)| \leq M.$$

In the following, we will use \mathbb{E}^j to denote the conditional expectation given $\mathbf{s}_1, \mathbf{s}_2, \dots$ except \mathbf{s}_j . It is indeed the expectation taken with respect to \mathbf{s}_j . And write

$$(3.31) \quad \text{center}^j(x) = x - \mathbb{E}^j(x),$$

where x is some random variable. We claim that

$$(3.32) \quad \mathbb{E} \left| n^{-1} \sum_{j>k} \text{center}^j(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \mathbf{s}_j) \right|^2 = O(1/(nv^2)),$$

$$(3.33) \quad \mathbb{E} \left| n^{-1} \sum_{j<k} \text{center}^j(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \mathbf{s}_j) \right|^2 = O(1/(nv^2))$$

and

$$(3.34) \quad \mathbb{E} \left| n^{-1} \sum_{j<k} \text{center}^j(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j) \right|^2 = O(1/(n^2v^2)).$$

Consider (3.32) first. Apparently by Lemma 8 we have

$$(3.35) \quad n^{-2} \sum_{j>k} \mathbb{E} \left| \text{center}^j(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \mathbf{s}_j) \right|^2 = O(1/(n^2v^3)).$$

Second, we also obtain

$$(3.36) \quad \begin{aligned} & n^{-2} \sum_{j_1 \neq j_2 > k} \mathbb{E} \left[\text{center}^{j_1}(\mathbf{s}_{j_1}^T \mathbf{A}_{kj_1}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \mathbf{s}_{j_1}) \right. \\ & \quad \left. \times \text{center}^{j_2}(\mathbf{s}_{j_2}^T \mathbf{A}_{kj_2}^{-1}(\bar{z}_1) \underline{\mathbf{A}}_k^{-1}(\bar{z}_2) \mathbf{s}_{j_2}) \right] = O(1/(n^2v^4)), \end{aligned}$$

which was ensured by the following estimates:

$$(3.37) \quad \mathbb{E} [\text{center}^{j_1}(\mathbf{s}_{j_1}^T \mathbf{A}_{kj_1}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{s}_{j_1}) \times \text{center}^{j_2}(\mathbf{s}_{j_2}^T \mathbf{A}_{kj_2}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{s}_{j_2})] = 0;$$

(remember the convention that z and \bar{z} are dropped from the corresponding expressions) and via (3.25), Hölder's inequality

$$\mathbb{E} |\text{center}^{j_1}(\mathbf{s}_{j_1}^T \mathbf{A}(k, j_2, j_1) \underline{\mathbf{A}}_k^{-1}(z_2) \mathbf{s}_{j_1}) \times \text{center}^{j_2}(\mathbf{s}_{j_2}^T \mathbf{A}(k, j_1, j_2) \underline{\mathbf{A}}_k^{-1}(\bar{z}_2) \mathbf{s}_{j_2})| \leq M/(n^2v^4),$$

where

$$\begin{aligned} \mathbf{A}(k, j_2, j_1) &= \mathbf{A}_{kj_1j_2}^{-1}(z_1) \mathbf{s}_{j_2} \mathbf{s}_{j_2}^T \mathbf{A}_{kj_1j_2}^{-1}(z_1) \beta_{kj_2j_1}(z), \\ \mathbf{A}_{kj_1j_2}^{-1}(z) &= (\mathbf{A} - \mathbf{s}_k \mathbf{s}_k^T - \mathbf{s}_{j_1} \mathbf{s}_{j_1}^T - \mathbf{s}_{j_2} \mathbf{s}_{j_2}^T - z \mathbf{I})^{-1}, \quad \beta_{kj_2j_1}(z) = (1 + \mathbf{s}_{j_2}^T \mathbf{A}_{kj_1j_2}^{-1}(z) \mathbf{s}_{j_2})^{-1}, \\ \text{and } \mathbf{A}(k, j_1, j_2) \text{ and } \beta_{kj_1j_2}(z) &\text{ are defined similarly. Thus (3.32) is true, as claimed.} \end{aligned}$$

Consider (3.33) next. Note that (3.35), (3.37) are still true if $\underline{\mathbf{A}}_k^{-1}(z_2)$ is replaced by $\underline{\mathbf{A}}_{kj_1j_2}^{-1}(z_2)$. Moreover, by (3.25), Lemma 7, Hölder's inequality we obtain

$$\begin{aligned} &\mathbb{E} \left| \text{center}^{j_1}(\mathbf{s}_{j_1}^T \mathbf{A}(k, j_2, j_1)(z_1) \underline{\mathbf{A}}(k, j_2, j_1)(z_2) \mathbf{s}_{j_1}) \right. \\ &\quad \left. \times \text{center}^{j_2}(\mathbf{s}_{j_2}^T \mathbf{A}(k, j_1, j_2)(\bar{z}_1) \underline{\mathbf{A}}(k, j_2, j_1)(\bar{z}_2) \mathbf{s}_{j_2}) \right| \\ &\leq \frac{M}{n^2} \left[\mathbb{E} \left\| \mathbf{s}_{j_2}^T \mathbf{A}_{kj_1j_2}^{-1}(z) \right\|^4 \left\| \mathbf{s}_{j_2}^T \underline{\mathbf{A}}_{kj_1j_2}^{-1}(z) \right\|^4 \left| \beta_{kj_2j_1}^2(z) \underline{\beta}_{kj_2j_1}^2(z) \right| \right] \\ &\quad \times \mathbb{E} \left\| \mathbf{s}_{j_1}^T \mathbf{A}_{kj_1j_2}^{-1}(z) \right\|^4 \left\| \mathbf{s}_{j_1}^T \underline{\mathbf{A}}_{kj_2j_1}^{-1}(z) \right\|^4 \left| \beta_{kj_1j_2}^2(z) \underline{\beta}_{kj_1j_2}^2(z) \right| \right]^{1/2} \leq M/(n^2 v^4), \end{aligned}$$

where $\underline{\mathbf{A}}_{kj_1j_2}^{-1}(z)$ is obtained from $\mathbf{A}_{kj_1j_2}^{-1}(z)$ with $\mathbf{s}_{k+1}, \dots, \mathbf{s}_n$ being replaced by $\underline{\mathbf{s}}_{k+1}, \dots, \underline{\mathbf{s}}_n$ and the remaining $\mathbf{s}_1, \dots, \mathbf{s}_{k-1}$ unchanged, $\underline{\beta}_{kj_1j_2}(z)$ is obtained from $\beta_{kj_1j_2}(z)$ with $\mathbf{A}_{kj_2j_1}^{-1}(z)$ replaced by $\underline{\mathbf{A}}_{kj_2j_1}^{-1}(z)$ and $\underline{\mathbf{A}}(k, j_2, j_1)(z)$ from $\mathbf{A}(k, j_2, j_1)(z)$ with $\mathbf{A}_{kj_2j_1}^{-1}(z)$ and $\beta_{kj_1j_2}(z)$, respectively, replaced by $\underline{\mathbf{A}}_{kj_2j_1}^{-1}(z)$ and $\underline{\beta}_{kj_1j_2}(z)$. Here $\underline{\mathbf{A}}(k, j_1, j_2)$ and $\underline{\beta}_{kj_2j_1}(z)$ can be similarly defined. These estimates imply (3.33).

Replacing $\underline{\mathbf{A}}_k^{-1}(z_2)$ by the identity matrix from (3.35)-(3.37) yields (3.34).

Let $u_n(z) = (z - (1 - n^{-1})b_{12}(z))^{-1}$. We now state the equality (2.9) in [3]

$$(3.38) \quad \mathbf{A}_k^{-1}(z) = -u_n(z) \mathbf{I} + b_{12}(z) B(z) + C(z) + D(z),$$

where

$$\begin{aligned} B(z) &= \sum_{j \neq k} u_n(z) (\mathbf{s}_j \mathbf{s}_j^T - n^{-1} \mathbf{I}) \mathbf{A}_{kj}^{-1}(z), \\ C(z) &= \sum_{j \neq k} (\beta_{kj}(z) - b_{12}(z)) u_n(z) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \end{aligned}$$

and

$$D(z) = n^{-1} b_{12}(z) u_n(z) \sum_{j \neq k} (\mathbf{A}_{kj}^{-1}(z) - \mathbf{A}_k^{-1}(z)).$$

Applying the definition of $C(z_1)$ and (3.26) gives

$$(3.39) \quad n^{-1} \mathbb{E}_k [\text{tr} C(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)] = C_1(z_1) + C_2(z_1),$$

where

$$C_1(z_1) = -b_{12}^2(z_1) n^{-1} \sum_{j \neq k} \mathbb{E}_k [\xi_{kj}(z_1) \mathbf{s}_j^T \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) \mathbf{s}_j]$$

and

$$C_2(z_1) = b_{12}^2(z_1) n^{-1} \sum_{j \neq k} \mathbb{E}_k [\beta_{kj}(z_1) \xi_{kj}^2(z) \mathbf{s}_j^T \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) \mathbf{s}_j].$$

Here

$$(3.40) \quad \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) = \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) u_n(z_1).$$

Define $\zeta_{kj3} = \text{center}^j(\mathbf{s}_j^T \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) \mathbf{s}_j)$.

We claim that the contribution from $C(z_1)$ is negligible. To verify it we distinguish two cases: $j > k$ and $j < k$. Consider $j > k$ first. From (3.29) and an estimate similar to (3.29) we have

$$(3.41) \quad \mathbb{E}|n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)|^4 \leq M(\mathbb{E}|n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{A}_{kj}^{-1}(\bar{z}_1)|^4 \mathbb{E}|n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-1}(z_2) \underline{\mathbf{A}}_k^{-1}(\bar{z}_2)|^4)^{1/2} \leq M/v^4.$$

where we use (3.24) as well. It follows from Lemma 8 and (3.41) that

$$\begin{aligned} \mathbb{E}|C_2(z_1)| &\leq Mn^{-1} \sum_{j \neq k} (\mathbb{E}|\xi_{kj}(z_1)|^4)^{1/2} [\mathbb{E}|\beta_{kj}(z_1)|^4 \\ &\quad \times (\mathbb{E}|\zeta_{kj3}|^4 + \mathbb{E}|n^{-1} \text{tr} \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2)|^4)]^{1/4} \leq M(nv^2)^{-1}. \end{aligned}$$

As for $C_1(z_1)$, write

$$(3.42) \quad \begin{aligned} &\mathbb{E}_k[\xi_{kj}(z_1) \mathbf{s}_j^T \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) \mathbf{s}_j] = \mathbb{E}_k(\eta_{kj}(z_1) \zeta_{kj3}) \\ &+ \mathbb{E}_k(n^{-1} \Gamma_{kj}(\text{tr} \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2) - \mathbb{E} \text{tr} \hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2))) + \mathbb{E}_k(\Gamma_{kj} n^{-1} \mathbb{E} \text{tr}(\hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2))). \end{aligned}$$

We conclude from (3.24), Lemmas 9 and 8 that the absolute moments of the first two terms above on the right hand have an order of $1/(nv^2)$. As for the last term, it was proved in Proposition 6.1 of [10] that

$$(3.43) \quad \mathbb{E}|n^{-1} \text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z)|^2 \leq \frac{M}{n^2 v^2 |z + c_n - 1 + 2c_n z m_n(z)|^2}$$

In view of (2.1), we have

$$(3.44) \quad z + c_n - 1 + 2c_n z m_n^0(z) = \sqrt{(a_n - z)(b_n - z)},$$

where $m_n^0(z)$ is obtained from $m(z)$ with c replaced by c_n , $a_n = (1 - \sqrt{c_n})^2$ and $b_n = (1 + \sqrt{c_n})^2$. From (2.10) we claim that

$$(3.45) \quad \frac{1}{h} \int_{a_l}^{a_r} \frac{|K((x-z)/h)|}{|z + c_n - 1 + 2c_n z m_n^0(z)|} du \leq M \frac{1}{h} \int_{a_l}^{a_r} \frac{|K((x-z)/h)|}{\sqrt{|(u-a_n)(b_n-u)|}} du \leq M.$$

Indeed, by a change of variables and dividing the integration region into $|q| \leq \delta$ and $|q| > \delta$ with $0 < \delta < \min\{\frac{b_n-x}{2}, \frac{x-a_n}{2}\}$, we have

$$\begin{aligned} &\frac{1}{h} \int_{a_l}^{a_r} \frac{|K((x-z)/h)|}{\sqrt{|(a_n-u)(b_n-u)|}} du = \frac{1}{h} \int_{x-a_r}^{x-a_l} I(|q| \leq \delta) \frac{|K(q/h + iv_0)|}{\sqrt{|(x-q-a_n)(b_n-(x-q))|}} dq \\ &\quad + \frac{1}{h} \int_{x-a_r}^{x-a_l} I(|q| > \delta) \frac{|K(q/h + iv_0)|}{\sqrt{|(x-q-a_n)(b_n-(x-q))|}} dq \\ &\leq M_x \frac{1}{h} \int_{x-a_r}^{x-a_l} |K(q/h + iv_0)| dq + \sup_{|q| > \delta} \left| \frac{q}{h} K\left(\frac{d}{h} + iv_0\right) \right| \frac{1}{\delta} \int_{a_l}^{a_r} \frac{1}{\sqrt{|(u-a_n)(b_n-u)|}} du \leq M, \end{aligned}$$

where M_x denotes some positive constant which depends on x . It follows from (3.43), (3.45), (3.24) and an inequality similar to (3.24) that

$$(3.46) \quad \mathbb{E} \left| \frac{1}{h} \int_{a_l}^{a_r} K\left(\frac{x - z_1}{h}\right) \mathbb{E}_k(\Gamma_{kj} n^{-1} \mathbb{E} \text{tr}(\hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2))) du_1 \right| \leq M/(nv^2),$$

where we also use the fact that $|n^{-1} \mathbb{E} \text{tr}(\hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2))| \leq M/v$.

For handling the case $j < k$, we define $\underline{\mathbf{A}}_{kj}^{-1}(z), \underline{\beta}_{kj}(z)$ and $\underline{\xi}_{kj}(z)$ by

$$\mathbf{s}_1, \dots, \mathbf{s}_{j-1}, \mathbf{s}_{j+1}, \dots, \mathbf{s}_{k-1}, \underline{\mathbf{s}}_{k+1}, \dots, \underline{\mathbf{s}}_n$$

as $\mathbf{A}_{kj}^{-1}(z), \beta_{kj}(z)$ and $\xi_{kj}(z)$ are defined by

$$\mathbf{s}_1, \dots, \mathbf{s}_{j-1}, \mathbf{s}_{j+1}, \dots, \mathbf{s}_{k-1}, \mathbf{s}_{k+1}, \dots, \mathbf{s}_n.$$

When $j < k$, similar to (3.23), we then decompose $\underline{\mathbf{A}}_k^{-1}(z_2)$ as

$$(3.47) \quad \underline{\mathbf{A}}_{kj}^{-1}(z_2) - \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_k^{-1}(z_2) \underline{\beta}_{kj}.$$

Note that \mathbf{s}_j is independent of $\underline{\mathbf{A}}_{kj}^{-1}(z_2)$. Apparently, the preceding argument for the case $j > k$ also works if we replace $\underline{\mathbf{A}}_k^{-1}(z_2)$ in $\hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2)$ with $\underline{\mathbf{A}}_{kj}^{-1}(z_2)$, the first term of (3.47), because the preceding argument used the independence between $\underline{\mathbf{A}}_k^{-1}(z_2)$ and \mathbf{s}_j when $j < k$. For another term of $C_2(z_1)$ due to the second term of (3.47), by (3.25), (3.29) and Lemma 8

$$\begin{aligned} & \mathbb{E} \left| \beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \xi_{kj}^2(z) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) u_n(z_1) \mathbf{s}_j \right| \\ & \leq Mv^{-1} (\mathbb{E} |\beta_{kj}(z_1)|^2 \mathbb{E} |\underline{\beta}_{kj}(z_2)|^2 \mathbb{E} |\xi_{kj}|^8 \mathbb{E} |\mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j u_n(z_1)|^4)^{1/4} \leq M/(nv^2). \end{aligned}$$

As for another term of $C_1(z_1)$, it follows from Holder's inequality, Lemmas 8, 9, (3.34), (3.24), (3.23) and (3.4) that yields

$$\begin{aligned} & n^{-1} \sum_{j < k} \mathbb{E}_k [\underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j u_n(z_1)] \\ (3.48) \quad & = n^{-1} \sum_{j < k} \mathbb{E}_k [\underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) u_n(z_1)] + A_1 \\ & = n^{-1} \mathbb{E} (n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2)) \sum_{j < k} \mathbb{E}_k [\underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) u_n(z_1)] + A_2 \\ & = n^{-1} u_n(z_1) b_{12}(z_2) \mathbb{E} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbb{E} (n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2)) \\ & \quad \times \sum_{j < k} \mathbb{E}_k [\eta_{kj}(z_1) + (n^{-1} \text{tr} \mathbf{A}^{-1}(z_1) - \mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z_1))] + A_3 \\ & = u_n(z_1) b_{12}(z_2) \mathbb{E} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbb{E} (n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2)) \\ & \quad \times (1 - k/n) E_k (n^{-1} \text{tr} \mathbf{A}^{-1}(z_1) - \mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z_1)) + A_4, \end{aligned}$$

where each A_j satisfies $E|A_j| \leq M/(nv^2)$, $j = 1, 2, 3, 4$ and the last term can be handled as in (3.46). Summarizing the above we have proved that

$$(3.49) \quad \mathbb{E} \left| \frac{1}{h} \int_{a_l}^{a_r} K\left(\frac{x - z_1}{h}\right) n^{-1} \text{tr} \mathbb{E}_k \left[C(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \right] du_1 \right| \leq \frac{M}{nv^2}.$$

Consider $D(z_1)$ now. When $j > k$ using (3.26) and recalling the definition of $\hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2)$ in (3.40) we obtain

$$n^{-1} \mathbb{E}_k \left[\text{tr} D(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \right] = n^{-2} b_{12}(z_1) \sum_{j \neq k} [D_1 + D_2]$$

where

$$D_1 = -n^{-1} \mathbb{E}_k \left[\text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \underline{\mathbf{A}}_{kj}^{-1}(z_1) \beta_{kj}(z_1) \right]$$

and

$$D_2 = \mathbb{E}_k \left[\beta_{kj}(z_1) \text{center}^j \left(\mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \underline{\mathbf{A}}_{kj}^{-1}(z_1) \mathbf{s}_j \right) \right].$$

By Lemmas 7, 8, Holder's inequality and (3.29) we have $\mathbb{E}|D_1| \leq M/v^2$ and $\mathbb{E}|D_2| \leq v^{-3/2}$. These imply that for $j > k$

$$(3.50) \quad \mathbb{E} |n^{-1} \text{tr} D(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)| \leq M/(nv^2).$$

When $j < k$, we resort to (3.47), the decomposition of $\underline{\mathbf{A}}_k^{-1}(z_2)$. As before, the above argument for the case $j > k$ also works for the term involving $\underline{\mathbf{A}}_{kj}^{-1}(z_2)$ if we replace $\underline{\mathbf{A}}_k^{-1}(z_2)$ with $\underline{\mathbf{A}}_{kj}^{-1}(z_2)$. Another term is

$$\frac{b_{12}(z_1)}{n^2} \sum_{j \neq k} \left[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \underline{\mathbf{A}}_{kj}^{-1}(z_1) \mathbf{s}_j u_n(z_1) \right],$$

which has, via (3.25), an order of $(nv^2)^{-1}$. Thus, the contribution from $C(z_1)$ and $D(z_1)$ is negligible.

Next consider $B(z_1)$. In view of (3.32) and (3.33) we may write

$$n^{-1} \text{tr} \mathbb{E}_k [B(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)] = B_1(z_1) + B_2(z_1) + A_5,$$

where

$$B_1(z_1) = -n^{-1} \sum_{j < k} \mathbb{E}_k \left[\underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \underline{\mathbf{A}}_{kj}^{-1}(z_1) \mathbf{s}_j u_n(z_1) \right],$$

$$B_2(z_1) = n^{-2} \sum_{j < k} \mathbb{E}_k \left[\underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \underline{\mathbf{A}}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j u_n(z_1) \right]$$

and $E|A_5| \leq \frac{M}{nv^2}$. By (3.25) and Lemma 8 we have $|B_2(z_1)| \leq M/(nv^2)$. With notation $\hat{\eta}_{kj} = \text{center}^j \left(\mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \right)$, from Lemma 8 we obtain

$$\mathbb{E} |(\hat{\eta}_{kj}) (n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_1) - \mathbb{E} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_1))| \leq M/(nv^2),$$

which, together with (3.33), implies that

$$\mathbb{E} \left| n^{-1} \sum_{j < k} (\hat{\eta}_{kj}) n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_1) \right| = O(n^{-1} v^{-2}).$$

Moreover, in view of Lemma 9 and (3.34) we have

$$\mathbb{E} \left| n^{-2} \sum_{j < k} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \eta_{kj} \right| \leq M/(nv^2).$$

Apparently by Lemma 8 we also have

$$\mathbb{E} |\hat{\eta}_{kj} \eta_{kj}| \leq M/(nv^2).$$

We then conclude from Lemmas 7, 8 and (3.26) that

$$(3.51) \quad \left| B_1(z_1) + n^{-3} b_{12}(z_2) u_n(z_1) \sum_{j < k} \mathbb{E}_k [\text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2)] \right| = A_6,$$

where $E|A_6| \leq \frac{M}{nv^2}$.

Furthermore by (3.24) we obtain

$$\begin{aligned} & n^{-3} \sum_{j < k} \mathbb{E}_k \left[\text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \right] \\ &= \frac{k-1}{n^3} \mathbb{E}_k \left[\text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \text{tr} \underline{\mathbf{A}}_k^{-1}(z_2) \right] + O\left(\frac{1}{nv^2}\right). \end{aligned}$$

It follows from Lemma 8, (3.43) and (3.45) that

$$\begin{aligned} & \mathbb{E}_k \left[\frac{k-1}{n^3} \text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \text{tr} \underline{\mathbf{A}}_k^{-1}(z_2) \right] \\ &= \frac{1}{n} \mathbb{E} \text{tr} \underline{\mathbf{A}}_k^{-1}(z_2) \mathbb{E}_k \left[\frac{k-1}{n^2} \text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \right] + A_7, \end{aligned}$$

where

$$(3.52) \quad \frac{1}{h} \int_{a_l}^{a_r} |K\left(\frac{x-z_1}{h}\right)| \mathbb{E} |A_7| du_1 \leq \frac{M}{nv^2}.$$

We then conclude that

$$(3.53) \quad B_1(z_1) + b_{12}(z_2) u_n(z_1) n^{-1} \mathbb{E} \text{tr} \underline{\mathbf{A}}_k^{-1}(z_2) \frac{k-1}{n^2} \mathbb{E}_k \left[\text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2) \right] =: A_8,$$

where A_8 satisfies (3.52) with A_7 replaced by A_8 .

Summarizing the argument from (3.39) to (3.53) yields

$$(3.54) \quad \begin{aligned} & n^{-1} \mathbb{E}_k [\text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)] = -n^{-1} u_n(z_1) \mathbb{E}_k [\text{tr} \underline{\mathbf{A}}_k^{-1}(z_2)] \\ & - u_n(z_1) b_{12}(z_1) b_{12}(z_2) n^{-1} \mathbb{E} (\text{tr} \mathbf{A}^{-1}(z_2)) \left[\frac{k-1}{n^2} \mathbb{E}_k (\text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2)) \right] + A_9, \end{aligned}$$

where A_9 satisfies (3.52) with A_7 replaced by A_9 .

By the formula (see (2.2) of [21]) $\underline{m}_n(z) = -z^{-1} n^{-1} \sum_{k=1}^n \beta_k(z)$, we have

$$(3.55) \quad \mathbb{E} \beta_1(z) = -z \mathbb{E} \underline{m}_n(z)$$

It follows from (3.5) and Lemma 8 that

$$|\mathbb{E} \beta_1(z) - b_1(z)| = |b_1(z)^2 \mathbb{E} (\beta_1(z) \xi_1^2(z))| \leq M/(nv)$$

and from (3.24) that

$$(3.56) \quad |b_1(z) - b_{12}(z)| \leq M/(nv).$$

These, together with (5.12) below, imply that

$$(3.57) \quad |b_{12}(z) + z\underline{m}_n^0(z)| \leq M/(nv).$$

This, along with (2.5), ensures that

$$(3.58) \quad n^{-1} \mathbb{E} \text{tr} \mathbf{A}^{-1}(z) = -\frac{c_n}{z + z\underline{m}_n^0(z)} + O\left(\frac{1}{nv}\right).$$

We then conclude from (3.54), (3.57), (3.58), Lemma 8 and (5.12) that

$$(3.59) \quad \begin{aligned} & n^{-1} \mathbb{E}_k \left[\text{tr} \mathbf{A}_k^{-1}(z_1) \mathbb{E}_k(\mathbf{A}_k^{-1}(z_2)) \right] \times \left[1 - \frac{k-1}{n} b_n(z_1, z_2) \right] \\ &= \frac{b_n(z_1, z_2)}{z_1 z_2 \underline{m}_n^0(z_1) \underline{m}_n^0(z_2)} + A_{10}, \end{aligned}$$

where A_{10} satisfies (3.52) with A_7 replaced by A_{10} and

$$b_n(z_1, z_2) = \frac{c_n \underline{m}_n^0(z_1) \underline{m}_n^0(z_2)}{(1 + \underline{m}_n^0(z_1))(1 + \underline{m}_n^0(z_2))}.$$

From (2.19) in [3] and the inequality above (6.37) in [11] we see that

$$(3.60) \quad \left| 1 - \frac{k-1}{n} b_n(z_1, z_2) \right| \geq Mv, \quad |1 - t b_n(z_1, z_2)| \geq Mv, \quad \text{for any } t \in [0, 1].$$

It follows that

$$(3.61) \quad \left| n^{-1} \sum_{k=1}^n \left(1 - \frac{k-1}{n} b_n(z_1, z_2) \right)^{-1} - \int_0^1 (1 - t b_n(z_1, z_2))^{-1} dt \right| \leq \frac{M}{nv^2}.$$

Similarly we have

$$\left| n^{-1} \sum_{k=1}^n \left(1 - \left| \frac{k-1}{n} b_n(z_1, z_2) \right| \right)^{-1} - \int_0^1 (1 - t |b_n(z_1, z_2)|)^{-1} dt \right| \leq \frac{M}{nv^2},$$

Moreover from Lemma 8 and (9.1)

$$\int_0^1 (1 - t |b_n(z_1, z_2)|)^{-1} dt = |b_n(z_1, z_2)|^{-1} \ln(1 - |b_n(z_1, z_2)|) = O(\ln 1/v).$$

It follows that

$$(3.62) \quad n^{-1} \sum_{k=1}^n \left| 1 - \frac{k-1}{n} b_n(z_1, z_2) \right|^{-1} = O(\ln 1/v).$$

We conclude from Lemma 8, (3.59), (3.61) and (3.62) that

$$\begin{aligned}
a_{n1}(z_1, z_2) &= b_n(z_1, z_2) n^{-1} \sum_{k=1}^n \left(1 - \frac{k-1}{n} b_n(z_1, z_2)\right)^{-1} + A_{11} \\
&= b_n(z_1, z_2) \int_0^1 (1 - t b_n(z_1, z_2))^{-1} dt + A_{12} \\
&= -\ln(1 - b_n(z_1, z_2)) + A_6 \\
(3.63) \quad &= -\ln\left((z_1 - z_2) \underline{m}_n^0(z_1) \underline{m}_n^0(z_2)\right) - \ln(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) + A_{12},
\end{aligned}$$

where A_{11} and A_{12} satisfy

$$(3.64) \quad \frac{1}{h} \int_{a_l}^{a_r} |K(\frac{x-z_1}{h})| \mathbb{E}|A_j| du_1 \leq \frac{M \ln 1/v}{nv^2}, \quad j = 5, 6$$

and in the last step one uses the fact that via (2.5)

$$z_1 - z_2 = \frac{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)}{\underline{m}_n^0(z_1) \underline{m}_n^0(z_2)} (1 - b_n(z_1, z_2)).$$

So far we have considered $z \in \gamma_2$, the top horizontal line. The above argument evidently works for the case of $z \in \gamma_1$, the bottom horizontal line, due to symmetry.

To deal with the cases when z belongs to two vertical lines of the contour, from Fubini's theorem and (2.9) we obtain for $j = 0, 1, 2$.

$$\int_{a_l}^{a_r} \left[\frac{1}{h} \int_0^{v_0} |K^{(j)}(\frac{x-u}{h} + iv)| dv \right] du = \int_0^{v_0} \left[\frac{1}{h} \int_{a_l}^{a_r} |K^{(j)}(\frac{x-u}{h} + iv)| du \right] dv < \infty.$$

This implies for $u \in [a_l, a_r]$

$$(3.65) \quad \frac{1}{h} \int_0^{v_0} |K^{(j)}(\frac{x-u}{h} + iv)| dv < \infty, \quad j = 0, 1, 2.$$

We also need the estimates (1.9a) and (1.9b) of [3], which hold under our truncation level. That is

$$(3.66) \quad \mathbb{P}(\|\mathbf{A}\| \geq \mu_1) = o(n^{-l}), \quad \mathbb{P}(\lambda_{\min}^{\mathbf{A}} \leq \mu_2) = o(n^{-l}),$$

for any $\mu_1 > (1 + \sqrt{c})^2$, $\mu_2 < (1 - \sqrt{c})^2$ and l . This implies that

$$(3.67) \quad \mathbb{P}(\|\mathbf{A}_k\| \geq \mu_1) = o(n^{-l}), \quad \mathbb{P}(\lambda_{\min}^{\mathbf{A}_k} \leq \mu_2) = o(n^{-l}).$$

Select a sequence of positive numbers ε_n satisfying for some $\beta \in (0, 1)$,

$$(3.68) \quad \varepsilon_n \downarrow 0, \quad \varepsilon_n \geq n^{-\beta}.$$

Then, as in [3], we introduce a truncation version of $X_n(z)$ on the top half parts of the two vertical lines of the contour as follows:

$$\hat{X}_n(z) = \begin{cases} X_n(z) & \text{for } u = a_r, v \in [n^{-1}\varepsilon_n, v_0h] \\ X_n(a_r + in^{-1}\varepsilon_n) & \text{for } u = a_r, v \in [0, n^{-1}\varepsilon_n] \\ X_n(a_l + in^{-1}\varepsilon_n) & \text{for } u = a_l, v \in [0, n^{-1}\varepsilon_n] \end{cases}$$

(one can similarly consider the bottom half parts of the two vertical lines). It follows that with probability one

$$\left| \int K\left(\frac{x-z}{h}\right)(\hat{X}_n(z) - \hat{X}_n(z))dz \right| \leq Mh\varepsilon_n \left(\frac{1}{a_r - \lambda_{\max}} + \frac{1}{\lambda_{\min} - a_l} \right) \rightarrow 0.$$

Indeed, there is an extra h above on the right hand which is needed in the proof of Theorem 1.

For $\hat{X}_n(z)$ on the two vertical lines $\gamma_2 \cup \gamma_4$, (3.11) is still true because there are at most finite number of points where the derivative of the corresponding truncation version of $\beta_k(z)$ do not exist. Moreover, for the truncation versions, the higher moments of $\mathbf{A}^{-1}(z)$, $\mathbf{A}_k^{-1}(z)$ and $\mathbf{A}_{kj}^{-1}(z)$ are bounded by (3.66) and (3.67) (see (3.1) in [3]). As pointed out in the paragraph below (3.2) in [3], the moments of $\beta_1(z)$, $\beta_{12}(z)$, $\beta^{\text{tr}}(z)$, $s_1^T \mathbf{A}_1^{-1}(z_1) \mathbf{T} \mathbf{A}_1^{-1}(z_2) \mathbf{s}_1$ are bounded as well. Using these facts, all the estimates holding for $z \in \gamma_1 \cup \gamma_2$ also holds for the case where $z \in \gamma_r \cup \gamma_l$. Note that the length of the vertical line is at most h . Via these facts, the arguments of the case $z \in \gamma_r \cup \gamma_l$, two vertical lines, can follow from those of the case $z \in \gamma_1 \cup \gamma_2$ (here we omit the details) and hence their limits have the same form as (3.63).

In the mean time, appealing to Cauchy's theorem gives

$$(3.69) \quad \frac{1}{h^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'\left(\frac{x_1 - z_1}{h}\right) K'\left(\frac{x_2 - z_2}{h}\right) \ln \left((z_1 - z_2) \underline{m}_n^0(z_1) \underline{m}_n^0(z_2) \right) dz_1 dz_2 = 0,$$

where the contour \mathcal{C}_2 is also a rectangle formed with four vertices $a_l - \varepsilon \pm 2iv_0h$ and $a_r + \varepsilon \pm 2iv_0h$ with $\varepsilon > 0$. One should note that the contour \mathcal{C}_2 encloses the contour \mathcal{C}_1 . Thus, in view of (3.63), it remains to find the limit of the following

$$(3.70) \quad - \frac{1}{2h^2\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'\left(\frac{x_1 - z_1}{h}\right) K'\left(\frac{x_2 - z_2}{h}\right) \ln(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) dz_1 dz_2,$$

which is done in Appendix 2.

4. THE CONVERGENCE RATE OF $\mathbb{E}m_n(z)$ TO $m(z)$

The aim of this section is to develop a sharp order for $\mathbb{E}\Gamma_1^2$ and $\mathbb{E}\Gamma_1^3$ which are crucial to the establishment of Theorem 1 with the stringent bandwidth restriction. Throughout this section, let $z = u + iv$ with $u \in [a, b]$ and $v \geq M_1/\sqrt{n}$ where M_1 is a sufficiently large positive constant.

We begin with a series of Lemmas.

Lemma 1. *Let*

$$g(z) = z + c_n - 1 + zc_n m_n^0(z) + zc_n \mathbb{E}m_n(z),$$

where $m_n^0(z)$ stands for the one obtained from $m(z)$ with c replaced by c_n . Then

$$(4.1) \quad |g(z)| \geq c_n v \mu_2 \mathbb{E}(n^{-1} \text{tr} \mathbf{A}^{-1}(z) \mathbf{A}^{-1}(\bar{z})) = c_n \mu_2 \Im[\mathbb{E}(n^{-1} \text{tr} \mathbf{A}^{-1}(z))]$$

and

$$(4.2) \quad |g(z)| \geq M\sqrt{v},$$

where $0 < \mu_2 < (1 - \sqrt{c_n})^2$.

Proof. It is straightforward to check that

$$(4.3) \quad \Im(z + c_n - 1 + z c_n m_n^0(z) + z c_n m_n(z)) \geq v + c_n v \lambda_{\min}(\mathbf{A}) n^{-1} \text{tr} \mathbf{A}^{-1}(z) \mathbf{A}^{-1}(\bar{z}).$$

Write

$$g(z) = z + c_n - 1 + z c_n m_n^0(z) + z c_n \mathbb{E}[m_n(z) I(D)] + z c_n \mathbb{E}[m_n(z) I(D^c)],$$

where the event $D = (\lambda_{\min}(\mathbf{A}) \leq \mu_2)$. Then (4.1) follows from (4.3) and (3.66). By (3.44)

$$(4.4) \quad |z + c_n - 1 + 2z c_n m_n^0(z)| \geq M \sqrt{v}.$$

On the other hand, it is proved in (6.109) of [10] that

$$(4.5) \quad |\mathbb{E} m_n(z) - s(z)| \leq \frac{M}{n v^{3/2}} \leq \rho_1 \sqrt{v},$$

where ρ_1 is sufficiently small. We then conclude from (4.4) and (4.5) that (4.2) holds. \square

Lemma 2.

$$(4.6) \quad |\mathbb{E} n^{-1} \text{tr} \mathbf{A}_1^{-2}(z)| \leq M/\sqrt{v}, \quad n^{-1} \sum_{k=1}^n |(\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \underline{\mathbf{A}}_k^{-1}(z))^2| \leq M/v,$$

$$(4.7) \quad n^{-1} \sum_{k=1}^n |\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-1}(z)| \leq \frac{M}{v |z + c_n - 1 + 2z c_n m_n^0(z)|},$$

$$(4.8) \quad n^{-1} \sum_{k=1}^n |\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \underline{\mathbf{A}}_k^{-1}(z) \mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-1}(z)| \leq M v^{-3/2},$$

and

$$(4.9) \quad |(ng(z))^{-1} \sum_{k=1}^n \mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z)| \leq \frac{M}{v^2 |z + c_n - 1 + 2z c_n m_n^0(z)|}.$$

Remark 3. From the derivation of (4.9) we see that the left side of the inequality of (4.9) multiplied by $g(z)$ is still less than the right side of the inequality.

Proof. Consider $\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z)$ first. When replacing $\underline{\mathbf{A}}_k^{-1}(z_2)$ by $\mathbf{A}_k^{-1}(z)$, the derivation in the last section for $\mathbb{E}_k(n^{-1} \text{tr} \mathbf{A}_k^{-1}(z_1) \underline{\mathbf{A}}_k^{-1}(z_2))$ also works for $\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z)$ except (3.42), (3.48) and the argument starting from (3.51). It is unnecessary to distinguish between the cases $j < k$ and $j > k$ in the current case and so we need not consider (3.42). By Lemma 8, (3.25) and (3.26), (3.48) reduces to

$$\begin{aligned} & n^{-1} \sum_j \mathbb{E} [\beta_{kj}(z) \xi_{kj}(z) \mathbf{s}_j^T \mathbf{A}_{kj}^{-2}(z) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j u_n(z)] \\ &= n^{-1} b_{12} u_n(z) \sum_{j < k} [\mathbb{E} (\eta_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-2} \mathbf{s}_j n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}) + \mathbb{E} (\Gamma_{kj} \Gamma_{kj}^{(2)}) \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}] + O(n^{-1} v^{-2}) \\ &= O(n^{-1} v^{-2}). \end{aligned}$$

Moreover, from Lemma 8, (3.51) turns out to be

$$(4.10) \quad \begin{aligned} & B_1(z) + n^{-3}b_{12}(z)u_n(z) \sum_j \mathbb{E}[\text{tr}\mathbf{A}_{kj}^{-2}(z)\text{tr}\mathbf{A}_{kj}^{-1}(z)] \\ &= B_1(z) + b_{12}(z)u_n(z)n^{-1}\mathbb{E}\text{tr}\mathbf{A}_1^{-1}(z)n^{-1}\mathbb{E}\text{tr}\mathbf{A}_1^{-2}(z) + O(n^{-1}v^{-2}). \end{aligned}$$

Therefore, as in (3.59), we have

$$(4.11) \quad n^{-1}\mathbb{E}[\text{tr}\mathbf{A}_k^{-2}(z)] \times [1 - b_n(z, z)] = \frac{b_n(z, z)}{(z\bar{m}_n^0(z))^2} + O(n^{-1}v^{-2}),$$

where $b_n(z, z)$ is obtained from $b_n(z_1, z_2)$ from (3.59) with $z_1 = z_2 = z$. From (2.5) and (2.4) one may verify that

$$(4.12) \quad 1 - b_n(z, z) = -(z + c_n - 1 + 2zc_n\bar{m}_n^0(z))\frac{\bar{m}_n^0(z)}{1 + \bar{m}_n^0(z)}.$$

It follows from (4.11), (4.12), (9.1), (4.4), (8.10) and Lemma 8 that

$$(4.13) \quad |n^{-1}\mathbb{E}\text{tr}\mathbf{A}_1^{-2}(z)| \leq M/\sqrt{v}.$$

As for the second inequality in (4.6), checking the above proof for $\mathbb{E}n^{-1}\text{tr}\mathbf{A}_1^{-2}(z)$ and the last section for $\mathbb{E}_k(n^{-1}\text{tr}\mathbf{A}_k^{-1}(z_1)\underline{\mathbf{A}}_k^{-1}(z_2))$ and referring to (3.59) we have

$$(4.14) \quad n^{-1}\mathbb{E}[\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1}] \times [1 - \frac{k-1}{n}b_n(z, z)] = \frac{b_n(z, z)}{(z\bar{m}_n^0(z))^2} + O(n^{-1}v^{-2}).$$

As in (3.61) and (3.62) we obtain

$$(4.15) \quad \begin{aligned} n^{-1} \sum_{k=1}^n (1 - \frac{k-1}{n}|b_n(z, z)|)^{-2} &= \int_0^1 (1 - t|b_n(z, z)|)^{-2} dt + O(\frac{1}{nv^3}) \\ &= \frac{|b_n(z, z)|}{1 - |b_n(z, z)|} + O(\frac{1}{nv^3}) = O(\frac{1}{v}) \end{aligned}$$

It follows from (4.14) and (4.15) that

$$(4.16) \quad n^{-1} \sum_{k=1}^n |(\mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-1}(z)\underline{\mathbf{A}}_k^{-1}(z))^2| = O(v^{-1}).$$

Consider (4.7) next. The strategy is to use (3.38). From (3.25) and Lemma 8 we obtain

$$\mathbb{E}[n^{-1}\text{tr}\underline{\mathbf{A}}_k^{-1}\mathbf{A}_k^{-1}D(z)] = \frac{b_{12}u_n(z)}{n^2} \sum_{j \neq k}^n \mathbb{E}[\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj}] = O(v^{-1}).$$

Apply (3.26) and (3.23) to write

$$(4.17) \quad \mathbb{E}[n^{-1}\text{tr}\underline{\mathbf{A}}_k^{-1}\mathbf{A}_k^{-1}C(z)] = n^{-1}u_n(z)b_{12} \sum_{j \neq k}^n \mathbb{E}[(\xi_{kj} + \beta_{12}(\xi_{kj})^2)\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} \mathbf{s}_j]$$

$$= \frac{u_n(z)b_{12}(z)}{n} \sum_{j < k}^n (C_3 + C_4 + C_5 + C_6 + C_7 + C_8) + \frac{u_n(z)b_{12}(z)}{n} \sum_{j > k}^n (C_9 + C_{10}),$$

where

$$\begin{aligned} C_3 &= \mathbb{E}[\eta_{kj} \text{center}^j (\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j) + \Gamma_{kj} (n^{-1} \text{tr} \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_{kj}^{-1} - \mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_{kj}^{-1})] \\ C_4 &= \mathbb{E}[\xi_{kj} (\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j)^2 \beta_{kj}], \quad C_5 = \mathbb{E}[\xi_{kj} (\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j)^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj} \beta_{kj}], \\ C_6 &= \mathbb{E}[\xi_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj}], \quad C_7 = \mathbb{E}[\beta_{kj} (\xi_{kj})^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j] \\ C_8 &= \mathbb{E}[\beta_{kj}^2 (\xi_{kj})^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j], \end{aligned}$$

and

$$C_9 = \mathbb{E}[\beta_{kj} \xi_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j], \quad C_{10} = \mathbb{E}[\beta_{kj}^2 \xi_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j].$$

It follows from Lemmas 8, 9, (3.26), (3.23) and (3.25) that $|C_j| \leq M/v, j = 3, 6, 7, 8, 9, 10$ and

$$C_j = (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \underline{\mathbf{A}}_k^{-1}(z))^2 C_{j1} + O(v^{-1}), \quad j = 4, 5,$$

where $|C_{j1}| \leq M/\sqrt{nv}$. Therefore

$$(4.18) \quad \mathbb{E}[n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} C(z)] = (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \underline{\mathbf{A}}_k^{-1}(z))^2 A_{13} + O(v^{-1}),$$

where $|A_{13}| \leq M/\sqrt{nv}$.

Next we use (3.26) and (3.23) to write

$$\begin{aligned} \mathbb{E}[n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} B(z)] &= n^{-1} u_n(z) \sum_{j \neq k}^n \mathbb{E}[\text{center}^j (\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} \mathbf{s}_j)] \\ &= \frac{u_n(z)}{n} \sum_{j < k}^n (B_3 + B_4 + B_5) + \frac{u_n(z)}{n} \sum_{j > k}^n B_6 \end{aligned}$$

where

$$\begin{aligned} B_3 &= \mathbb{E}[(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j)^2 \beta_{kj} - n^{-1} \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j \beta_{kj}] \\ B_4 &= \mathbb{E}[(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j)^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj} - n^{-1} \mathbf{s}_j^T \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{s}_j \beta_{kj} \beta_{kj}], \\ B_5 &= -\mathbb{E}[\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj} - n^{-1} \mathbf{s}_j^T \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_{kj}^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj}] \end{aligned}$$

and

$$B_6 = -\mathbb{E}(\mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj} - n^{-1} \mathbf{s}_j^T \mathbf{A}_{kj}^{-2} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_{kj}^{-1} \mathbf{s}_j \beta_{kj}).$$

In view of Lemmas 8, 9, (3.26), (3.24) and (3.25) we have

$$B_j = (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1})^2 B_{j1} + O(v^{-1}), \quad j = 3, 4,$$

where $|B_{j1}| \leq M$ and

$$B_j = -b_{12} \mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-2} \underline{\mathbf{A}}_k^{-1}) + O(v^{-1}), \quad j = 5, 6.$$

Thus

$$(4.19) \quad \begin{aligned} \mathbb{E}(n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-1} \mathbf{A}_k^{-1} B(z)) &= (\mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1}))^2 A_{14} \\ &\quad - b_{12} u_n(z) \mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-2} \underline{\mathbf{A}}_k^{-1}) + O(v^{-1}), \end{aligned}$$

where $|A_{14}| \leq M$.

Note that the coefficient of $\mathbb{E}(n^{-1}\text{tr}\mathbf{A}_k^{-2}\underline{\mathbf{A}}_k^{-1})$ in (4.19) is the same as that of $\mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-2}$ in (4.10). Summarizing the above we have thus obtained

$$(4.20) \quad \mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-2}\underline{\mathbf{A}}_k^{-1}(1-b_n(z, z)) = -u_n\mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1} + (\mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1})^2 A_{15} + O(v^{-1}),$$

where $|A_{15}| \leq M$. This, together with (4.6), implies (4.7).

As for (4.8), in view of (4.20) we have

$$(4.21) \quad (1-b_n(z, z))\mathbb{E}(\frac{1}{n}\text{tr}\mathbf{A}_k^{-2}\underline{\mathbf{A}}_k^{-1})\mathbb{E}(n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1}) = a_1(z)[\mathbb{E}(n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1})]^2 \\ + a_2(z)[\mathbb{E}(n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1})]^3 + a_3(z)\mathbb{E}(n^{-1}\text{tr}\mathbf{A}_k^{-1}\underline{\mathbf{A}}_k^{-1}),$$

where $|a_1(z)| \leq M$, $|a_2(z)| \leq M$ and $|a_3(z)| \leq M/v$. Moreover we conclude from (3.60), (4.15), (4.12) and Lemma 1 that

$$(4.22) \quad n^{-1} \sum_{k=1}^n (1 - \frac{k-1}{n} |b_n(z, z)|)^{-3} = \int_0^1 (1 - t|b_n(z, z)|)^{-3} dt + O(v^{-1}) \\ = (1 - |b_n(z, z)|)^{-2} + O(v^{-1}) = O(v^{-1}).$$

Then (4.8) follows from (4.14), (4.21), (4.22), and (4.12).

To establish (4.9), we observe that Lemmas 1 and 8 imply

$$(4.23) \quad |\mathbb{E}n^{-1}\text{tr}\mathbf{A}^{-1}(z)\mathbf{A}^{-1}(\bar{z})|/|g(z)| \leq M/v, \\ \mathbb{E}|n^{-1}\text{tr}\mathbf{A}^{-1}(z)\mathbf{A}^{-1}(\bar{z})|^k/|g(z)| \leq M(v^k|g(z)|)^{-1} [E|\Gamma|^k + |\Im(\mathbb{E}n^{-1}\text{tr}\mathbf{A}^{-1})|^k] \\ \leq M/v^k, \quad k = 2, 4, 8.$$

This key fact implies that whenever

$$\mathbb{E}n^{-1}\text{tr}\mathbf{A}^{-1}(z)\mathbf{A}^{-1}(\bar{z}) \text{ and } \mathbb{E}|n^{-1}\text{tr}\mathbf{A}^{-1}(z)\mathbf{A}^{-1}(\bar{z})|^k, \quad k = 2, 4, 8$$

appear, dividing them by $|g(z)|$ does not change their original sizes. As consequences of this fact, applying (4.23) and (3.24) then ensures that (8.1) and (8.2) are still true when we divide the expectations in them by $|g(z)|$. For example, by (4.23) and (3.24) we have for $m = 2, 4, 6, 8$

$$(4.24) \quad |g(z)|^{-1}\mathbb{E}|\eta_{kj}|^m \leq M(n^{m/2}|g(z)|)^{-1}\mathbb{E}|n^{-1}\text{tr}\mathbf{A}^{-1}(z)\mathbf{A}^{-1}(\bar{z})|^{\frac{m}{2}} \\ + M(n^{m/2}v^{m/2})^{-1} \leq M(n^{m/2}v^{m/2})^{-1}.$$

We also provide an argument in Lemma 9 for (8.1) when the expectation in it is divided by $|g(z)|$. Moreover by (4.23) and (3.24) one may verify that the first three conclusions of Lemma 9 are still true when the expectations in them are divided by $|g(z)|$ as in the last claim of Lemma 9. From now on until the end of this lemma we mean the corresponding expressions divided by $|g(z)|$ whenever we quote (8.1), (8.2) and Lemma 9.

Now we resort to use (3.38) again. From (4.23), (3.26) and (3.25), Lemmas 8 and 1 we have

$$\begin{aligned} |g(z)|^{-1} |\mathbb{E}[n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-2} \mathbf{A}_k^{-1} D(z)]| &\leq M(v^3 n^2 |g(z)|)^{-1} \sum_{j \neq k}^n \mathbb{E}(\|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}\|^2 |\beta_{kj}|) \\ &\leq M(v^3 n^2 |g(z)|)^{-1} \sum_{j \neq k}^n [\mathbb{E} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) |b_{12}| + \mathbb{E}(\|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}\|^2 |b_{12} \beta_{kj} \xi_{kj}|)] \leq M/v^2, \end{aligned}$$

where $\|\cdot\|$ denotes the spectral norm or Euclidean norm of matrices or vectors. As in (4.18) and (4.19), by (8.1), (3.24), (8.2), (3.26) and Lemmas 9, 8 one may verify that

$$\begin{aligned} g(z)^{-1} \mathbb{E}[n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-2} \mathbf{A}_k^{-1} C(z)] &= (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1})^2 A_{16} \\ &\quad + \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1}) \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-2}) A_{17} + O(v^{-2}), \end{aligned}$$

and that

$$\begin{aligned} g(z)^{-1} \mathbb{E}[n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-2} \mathbf{A}_k^{-1} B(z)] &= (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1})^2 A_{18} + \\ &\mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1}) \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-2}) A_{19} - b_{12} u_n(z) \mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-2} \underline{\mathbf{A}}_k^{-2}) + O(v^{-2}), \end{aligned}$$

where $|A_{16}| \leq M$, $|A_{17}| \leq M/\sqrt{nv}$, $|A_{18}| \leq M/\sqrt{v}$ and $|A_{19}| \leq M$. These imply that

$$\begin{aligned} g(z)^{-1} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-2} \underline{\mathbf{A}}_k^{-2}) (1 - b_n(z, z)) &= -u_n(z) g(z)^{-1} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-2}) \\ &\quad + (\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1})^2 A_{20} + \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-1}) \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-1} \underline{\mathbf{A}}_k^{-2}) A_{21} + O(v^{-2}), \end{aligned}$$

where $|A_{20}| \leq M/\sqrt{v}$ and $|A_{21}| \leq M$. Thus (4.9) follows from (4.6), (4.7) and (4.8) immediately. \square

Lemma 3.

$$(4.25) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_j - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_j|^2 \leq M/(nv^2),$$

$$(4.26) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_j - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_j|^2 \leq M/(nv^4),$$

$$(4.27) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i|^4 \leq M/(nv^2)$$

and

$$(4.28) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_i - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_i|^4 \leq M/(nv^6),$$

where \mathbf{e}_i is the p -dimensional vector with the i -th coordinate being 1 and the remaining being zero.

Proof. Consider $i = j$ first. Let $\Phi_1(z) = \mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i$. By (3.23) and (3.26) write

$$\begin{aligned} \Phi_1(z) &= \sum_{k=2}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) (\mathbf{e}_i^T (\mathbf{A}_1^{-1}(z) - \mathbf{A}_{1k}^{-1}(z)) \mathbf{e}_i) \\ (4.29) \quad &= - \sum_{k=2}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) (\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i \mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k \beta_{1k}). \end{aligned}$$

Let

$$\gamma_{ke} = \gamma_{rke} - n^{-1} \mathbf{e}_i^T \mathbf{A}_{1k}^{-2}(z) \mathbf{e}_i, \quad \gamma_{rke} = \mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i \mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k.$$

From Lemmas 7 and 8 we obtain

$$\begin{aligned} \mathbb{E}|\gamma_{rke}\beta_{1k}|^4 &\leq M(\mathbb{E}|\beta_{1k}|^8)^{1/2}(\mathbb{E}|\gamma_{ke}|^8)^{1/2} + M(\mathbb{E}|\beta_{1k}|^8)^{1/2}(\mathbb{E}|n^{-1}\mathbf{e}_i^T \mathbf{A}_{1k}^{-2}(z) \mathbf{e}_i|^8)^{1/2} \\ (4.30) \quad &\leq M(n^4 v^6)^{-1}(1 + (\mathbb{E}|\Phi_1(z)|^4)^{1/2}), \end{aligned}$$

where we also use the facts that

$$\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{A}_{1k}^{-1}(\bar{z}) \mathbf{e}_i = v^{-1} \Im(\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i)$$

and that via (3.23), Lemmas 7 and 8

$$(4.31) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i - \mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i|^4 \leq M(nv^2)^{-1}.$$

By Lemma 7, estimates similar to (3.6) and (3.14) we have

$$\begin{aligned} \mathbb{E}_{k-1} |\gamma_{rke}|^2 &\leq \frac{M}{n^2 v^2} \mathbb{E}_{k-1} |\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^2, \\ \mathbb{E}_{k-1} |\gamma_{rke}(\beta_{1k}^{tr})^2 \eta_{1k}|^2 &\leq \frac{M}{n^3 v^3} \left(\mathbb{E}_{k-1} |\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^4 \right)^{1/2} \left(\mathbb{E}_{k-1} |\beta_{1k}^{tr}|^4 \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{k-1} |\gamma_{rke} \beta_{1k} (\beta_{1k}^{tr})^2 \eta_{1k}^2|^2 \\ &\leq \frac{M}{v^2} \mathbb{E}_{k-1} |\gamma_{rke} \beta_{1k}^{tr} \eta_{1k}^2|^2 \leq \frac{M}{n^4 v^6} \left(\mathbb{E}_{k-1} |\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^4 \right)^{1/2} \left(\mathbb{E}_{k-1} |\beta_{1k}^{tr}|^2 \right)^{1/2}. \end{aligned}$$

These, together with $\beta_{1k} = \beta_{1k}^{tr} - (\beta_{1k}^{tr})^2 \eta_{1k} + \beta_{1k} (\beta_{1k}^{tr})^2 \eta_{1k}^2$, imply that

$$(4.32) \quad \mathbb{E}_{k-1} |\gamma_{rke} \beta_{1k}|^2 \leq \frac{M}{n^2 v^2} \left(\mathbb{E}_{k-1} |\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^4 \right)^{1/2} \left[\left(\mathbb{E}_{k-1} |\beta_{1k}^{tr}|^4 \right)^{1/2} + 1 \right].$$

It follows from (4.30), (4.32), (4.31), Burkholder's inequality and Lemma 8 that

$$\begin{aligned} E|\Phi_1(z)|^4 &\leq ME \left(\sum_{k=2}^p \mathbb{E}_{k-1} |\gamma_{rke} \beta_{1k}|^2 \right)^2 + M \sum_{k=2}^p E|\gamma_{rke} \beta_{1k}|^4 \\ &\leq \frac{M}{nv^2} (E|\Phi_1(z)|^4)^{1/2} + \frac{M}{nv^2}. \end{aligned}$$

Solving the inequality yields (4.27).

As for $i \neq j$, from (4.27), (4.31) and Burkholder's inequality we obtain

$$\mathbb{E}|\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^4 \leq \frac{M}{n^2} E|\mathbf{e}_i^T \mathbf{A}_{1k}^{-1}(z) \mathbf{A}_{1k}^{-1}(\bar{z}) \mathbf{e}_i|^2 \leq \frac{M}{n^2 v^2} E|\mathbf{e}_i^T \mathbf{A}_1^{-1}(z) \mathbf{e}_i|^2 \leq \frac{M}{n^2 v^2}$$

and

$$\mathbb{E}|\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^8 \leq \frac{M}{n^4 v^6}.$$

These ensure that

$$\mathbb{E}|\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i \mathbf{e}_j^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k|^2 \leq M(\mathbb{E}|\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^4 \mathbb{E}|\mathbf{e}_j^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k|^4)^{1/2} \leq M(n^2 v^2)^{-1}$$

and

$$\mathbb{E}|\xi_{1k} \beta_{1k} \mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i \mathbf{e}_j^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k|^2$$

$$\leq M(\mathbb{E}|\mathbf{s}_k^T \mathbf{A}_{1k}^{-1}(z) \mathbf{e}_i|^8 \mathbb{E}|\mathbf{e}_j^T \mathbf{A}_{1k}^{-1}(z) \mathbf{s}_k|^8)^{1/4} (\mathbb{E}|\xi_{1k}|^8 \mathbb{E}|\beta_{1k}|^8)^{1/4} \leq M(n^3 v^4)^{-1}.$$

By the above two estimates, (3.26) and replacing \mathbf{e}_i of (4.29) by \mathbf{e}_j we obtain (4.25).

In view of Cauchy's theorem we have

$$(4.33) \quad \mathbb{E}|\mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_j - \mathbb{E} \mathbf{e}_i^T \mathbf{A}_1^{-2}(z) \mathbf{e}_j|^2 \leq M v^{-2} \sup_{\zeta \in \Gamma_\zeta} \mathbb{E}|\Phi_1(\zeta)|^2,$$

where $\Gamma_\zeta = \{\zeta : |\zeta - z| = v/2\}$. This, together with (4.25), ensures (4.26). Similarly from (4.27) we can obtain (4.28). □

Lemma 4.

$$(4.34) \quad |\mathbb{E}(\eta_1)^3| \leq M/(n^{\frac{3}{2}}v),$$

$$(4.35) \quad |\mathbb{E}[\mathbb{E}_k(\eta_k^{(2)}) \mathbb{E}_k(\eta_k^{(2)} \eta_k)]| \leq M/(nv^2),$$

Proof. For $m = 1, 2, 3$, write

$$\mathbf{s}_k^T \mathbf{B}_m \mathbf{s}_k - n^{-1} \text{tr} \mathbf{B}_1 = \sum_{i=1}^p (X_{ki}^2 - 1) (\mathbf{B}_m)_{ii} + \sum_{i \neq j} X_{ki} X_{kj} (\mathbf{B}_m)_{ij},$$

where $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ are symmetric, independent of \mathbf{s}_k . A direct calculation then yields

$$(4.36) \quad \mathbb{E} \left[\prod_{m=1}^3 (\mathbf{s}_k^T \mathbf{B}_m \mathbf{s}_k - n^{-1} \text{tr} \mathbf{B}_m) \right] = n^{-3} \mathbb{E}(X_{11}^2 - 1)^3 \sum_{i=1}^p \mathbb{E} \left[\prod_{m=1}^3 (\mathbf{B}_m)_{ii} \right]$$

$$(4.37) \quad + 2n^{-3} (\mathbb{E} X_{11}^3)^2 \sum_{m_1, m_2, m_3} \sum_{i_1 \neq i_2} \mathbb{E}[(\mathbf{B}_{m_1})_{i_1 i_1} (\mathbf{B}_{m_2})_{i_2 i_2} (\mathbf{B}_{m_3})_{i_1 i_2}]$$

$$(4.38) \quad + 4n^{-3} (\mathbb{E} X_{11}^4 - 1) \sum_{m_1, m_2, m_3} \sum_{i_1 \neq i_2} \mathbb{E}[(\mathbf{B}_{m_1})_{i_1 i_1} (\mathbf{B}_{m_2})_{i_1 i_2} (\mathbf{B}_{m_3})_{i_1 i_2}]$$

$$(4.39) \quad + 4n^{-3} (\mathbb{E} X_{11}^3)^2 \sum_{i_1 \neq i_2} \mathbb{E}[(\mathbf{B}_1)_{i_1 i_2} (\mathbf{B}_2)_{i_1 i_2} (\mathbf{B}_3)_{i_1 i_2}]$$

$$(4.40) \quad + 8n^{-3} \sum_{i_1 \neq i_2, i_2 \neq i_3, i_1 \neq i_3} \mathbb{E}[(\mathbf{B}_1)_{i_1 i_2} (\mathbf{B}_2)_{i_1 i_3} (\mathbf{B}_3)_{i_2 i_3}].$$

where each m_i runs over 1, 2, 3, $m_i \neq m_j$ for any $i \neq j$.

Consider $\mathbb{E}(\eta_1)^3$ now. In this case $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \mathbf{A}_1^{-1}(z)$ in (4.36). Note that $\mathbb{E}(\mathbf{A}_1^{-1}(z))_{ii} = n^{-1} \mathbb{E} \text{tr} \mathbf{A}_1^{-1}(z)$ is bounded. By Lemma 3

$$(4.41) \quad n^{-3} \mathbb{E}(X_{11}^2 - 1)^3 \sum_{i=1}^p \mathbb{E} \left[\prod_{m=1}^3 (\mathbf{B}_m)_{ii} \right] = O(n^{-2}).$$

From Lemma 3 and Hölder's inequality we also have

$$n^{-3} \left| \sum_{i_1 \neq i_2} \mathbb{E}[(\mathbf{B}_{m_1})_{i_1 i_1} (\mathbf{B}_{m_2})_{i_2 i_2} (\mathbf{B}_{m_3})_{i_1 i_2}] \right| \leq M n^{-3/2} (n^{-1} \mathbb{E} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{A}_1^{-1}(\bar{z}))^{1/2},$$

which implies

$$(4.42) \quad (4.37) = O(n^{-3/2} v^{-1}).$$

Similarly, one may verify that

$$(4.43) \quad (4.38) = O(n^{-2} v^{-1}), \quad (4.39) = O(n^{-2} v^{-2}).$$

This, together with Lemma 3, implies that

$$(4.40) = 8n^{-3} \sum_{i_1=1}^p \sum_{i_2=1}^p \sum_{i_3=1}^p \mathbb{E}[(\mathbf{A}_1^{-1}(z))_{i_1 i_2} (\mathbf{A}_1^{-1}(z))_{i_1 i_3} (\mathbf{A}_1^{-1}(z))_{i_2 i_3}] + O(n^{-2} v^{-2})$$

$$(4.44) \quad = 8n^{-3} \sum_{i=1}^p \mathbb{E}(\mathbf{e}_i^T \mathbf{A}_1^{-3}(z) \mathbf{e}_i) + O(n^{-2} v^{-2}) = O(n^{-2} v^{-2}).$$

Thus (4.34) follows from (4.41)-(4.44).

Consider (4.35) next. Write

$$\mathbb{E}[\mathbb{E}_k(\eta_k^{(2)}) \mathbb{E}_k(\eta_k^{(2)} \eta_k)] = \mathbb{E}[\underline{\eta}_k^{(2)} \eta_k^{(2)} \eta_k],$$

where $\underline{\eta}_k^{(2)} = \mathbf{s}_k^T \underline{\mathbf{A}}_k^{-2}(z) \mathbf{s}_k - n^{-1} \text{tr} \underline{\mathbf{A}}_k^{-2}(z)$. In this case $\mathbf{B}_1 = \underline{\mathbf{A}}_k^{-2}(z)$, $\mathbf{B}_2 = \mathbf{A}_k^{-2}(z)$ and $\mathbf{B}_3 = \mathbf{A}_k^{-1}(z)$ in (4.36). Applying Lemmas 2 and 3 we have

$$(4.45) \quad n^{-3} \mathbb{E}(X_{ki}^2 - 1)^3 \sum_{i=1}^p \mathbb{E} \left[\prod_{m=1}^3 (\mathbf{B}_m)_{ii} \right] = O(n^{-2} v^{-2}).$$

Similarly one may verify that

$$(4.46) \quad (4.37) = O(n^{-3/2} v^{-3}), \quad (4.38) = O(n^{-2} v^{-3}), \quad (4.39) = O(n^{-2} v^{-4}),$$

where we use the fact that $\mathbb{E} n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) \mathbf{A}_k^{-2}(\bar{z}) \leq 1/v^3$. As in (4.44) we obtain

$$(4.40) = 8n^{-3} \sum_{i=1}^p \mathbb{E}(\mathbf{e}_i^T \underline{\mathbf{A}}_1^{-2}(z) \mathbf{A}_1^{-3}(z) \mathbf{e}_i) + O(n^{-2} v^{-4}) = O(n^{-2} v^{-4}).$$

These imply (4.35). □

Lemma 5.

$$n^{-2} \mathbb{E}(\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z))^2 = n^{-4} b_1^2(z) \sum_{k=1}^n \mathbb{E}[\text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z)] + O(n^{-2} v^{-2}).$$

Proof. Write

$$(4.47) \quad \beta_k = \beta_k^{\text{tr}} - \beta_k \beta_k^{\text{tr}} \eta_k(z).$$

Let

$$u_k = (\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k).$$

Applying (4.47) twice and referring to (3.9) we then have

$$\mathbb{E}[\text{tr}\mathbf{A}^{-1}(z) - \mathbb{E}\text{tr}\mathbf{A}^{-1}(z)]^2 = \sum_{k=1}^n \mathbb{E}(u_k)^2 = \sum_{k=1}^n \mathbb{E}(q_1 + q_2 + q_3 + q_4 + q_5 + q_6),$$

where

$$\begin{aligned} q_1 &= [\mathbb{E}_k(\beta_k^{\text{tr}}\eta_k^{(2)})]^2, \quad q_2 = [(\mathbb{E}_k - \mathbb{E}_{k-1})((\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k)]^2, \\ q_3 &= [(\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2)]^2 \\ q_4 &= -2\mathbb{E}_k(\beta_k^{\text{tr}}\eta_k^{(2)})(\mathbb{E}_k - \mathbb{E}_{k-1})((\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k), \\ q_5 &= -2(\mathbb{E}_k - \mathbb{E}_{k-1})((\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k)(\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2), \\ \text{and} \\ q_6 &= 2\mathbb{E}_k(\beta_k^{\text{tr}}\eta_k^{(2)})(\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2). \end{aligned}$$

It follows from (3.7) and (3.15) that

$$|n^{-2} \sum_{k=1}^n \mathbb{E}q_3| \leq Mn^{-2}v^{-2} \sum_{k=1}^n \mathbb{E}|(\beta_k^{\text{tr}})^2 \eta_k^2(z)|^2 \leq M/(n^2v^2).$$

Similar to (3.15), one may verify that

$$(4.48) \quad \mathbb{E}|\beta_k^{\text{tr}}\eta_k^{(2)}|^8 \leq M/(n^4v^{12}).$$

We then conclude from (3.14), (3.15) (4.48), Lemmas 2 and 8 and Hölder's inequality that

$$\begin{aligned} \mathbb{E}|(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k|^4 &\leq M\mathbb{E}|(\beta_k^{\text{tr}})^2 \eta_k^{(2)} \eta_k|^4 \\ (4.49) \quad &+ M|\mathbb{E}n^{-1}\text{tr}\mathbf{A}_k^{-2}(z)|^4 \mathbb{E}|(\beta_k^{\text{tr}})^2 \eta_k|^4 + M\mathbb{E}|\beta_k^{\text{tr}}\Gamma_k^{(2)} \eta_k|^4 \leq M/(n^2v^4). \end{aligned}$$

Similarly we have

$$(4.50) \quad \mathbb{E}|\beta_k(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2|^2 \leq M/(n^2v^3).$$

In view of (4.49) and (4.50)

$$\mathbb{E}q_5 = O(n^{-1}v^{-2}), \quad \mathbb{E}q_2 = O(n^{-1}v^{-2}).$$

As in (4.49) by (3.6), (3.14), Lemma 8 we have

$$\mathbb{E}|(\beta_k^{\text{tr}})^3 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2|^2 \leq M/(n^2v^3).$$

By (3.7) and (3.14) we obtain

$$(4.51) \quad \mathbb{E}|\beta_k(\beta_k^{\text{tr}})^3 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^3|^2 \leq M/(n^3v^5).$$

These two estimates, together with (4.47) and (4.48), ensure that

$$(4.52) \quad \mathbb{E}q_6 = O(n^{-1}v^{-2}).$$

Write

$$(4.53) \quad \beta_k^{\text{tr}} = b_1 - b_1^2 \Gamma_k + \beta_k^{\text{tr}} b_1^2 (\Gamma_k)^2.$$

In view of (3.20), (4.53), Lemmas 9, 2 and 8 we may write

$$\begin{aligned}
n^{-2} \sum_{k=1}^n \mathbb{E} q_1 &= n^{-4} b_1^2(z) \sum_{k=1}^n \mathbb{E} [\text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z)] \\
&+ 2n^{-3} b_1^3(z) \sum_{k=1}^n \mathbb{E} [(n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z) - n^{-1} \mathbb{E}(\text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z))) \Gamma_k] + O(n^{-2} v^{-2}) \\
&= n^{-4} b_1^2(z) \sum_{k=1}^n \mathbb{E} [\text{tr} \mathbf{A}_k^{-2}(z) \underline{\mathbf{A}}_k^{-2}(z)] + O(n^{-2} v^{-2}).
\end{aligned}$$

Likewise, by (3.20), (4.53), (4.35) and Lemmas 2, 9 and 8 we have

$$\begin{aligned}
n^{-2} \sum_{k=1}^n \mathbb{E} q_4 &= -2n^{-2} b_1^3 \sum_{k=1}^n \mathbb{E} [\mathbb{E}_k(\eta_k^{(2)}) (\mathbb{E}_k - \mathbb{E}_{k-1}) (\mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k)] + O(n^{-2} v^{-2}) \\
&= -2n^{-2} b_1^3 \sum_{k=1}^n \mathbb{E} [\mathbb{E}_k(\eta_k^{(2)}) \mathbb{E}_k(\eta_k^{(2)} \eta_k)] + 2n^{-3} b_1^3 \sum_{k=1}^n \mathbb{E} [n^{-1} \text{tr} \mathbf{A}_k^{-2}(z) n^{-1} \text{tr} \mathbf{A}_k^{-1}(z) \underline{\mathbf{A}}_k^{-2}(z)] \\
&\quad + O(n^{-2} v^{-2}) = O(n^{-2} v^{-2}).
\end{aligned}$$

□

Lemma 6.

$$n^{-3} |\mathbb{E}(\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z))|^3 \leq M/(n^2 v^2).$$

Proof. A direct calculation indicates that

$$(4.54) \quad \mathbb{E} [\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)]^3 = \sum_{k=1}^n \mathbb{E} (u_k)^3 + 3 \sum_{k_1 \neq k_2}^n \mathbb{E} (u_{k_1}^2 u_{k_2}).$$

Referring to the expressions of q_1, q_2, q_3 in the last lemma we have

$$\begin{aligned}
n^{-3} \left| \sum_{k=1}^n \mathbb{E} [(\mathbb{E}_k - \mathbb{E}_{k-1}) (\beta_k \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k)]^3 \right| &\leq M n^{-3} \sum_{k=1}^n [\mathbb{E} |\beta_k^{\text{tr}} \eta_k^{(2)}|^3] \\
&+ \mathbb{E} |(\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k|^3 + \mathbb{E} |\beta_k (\beta_k^{\text{tr}})^2 \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \eta_k^2|^3 \leq M/(n^2 v^{3/2}),
\end{aligned}$$

where the estimates can be obtained as in (4.48), (4.49) and (4.51).

When $k_1 < k_2$,

$$\mathbb{E}(u_{k_1}^2 u_{k_2}) = 0.$$

When $k_1 > k_2$ we have

$$(4.55) \quad \mathbb{E}(u_{k_1}^2 u_{k_2}) = \mathbb{E}[u_{k_2} E_{k_2}(u_{k_1}^2)].$$

Here we reminder that the term $\mathbb{E}_{k_2}(u_{k_1}^2)$ is similar to $\mathbb{E}(u_k^2)$, given in Lemma 5, except that the former is the conditional expectation and the later is the expectation. By the expressions of $q_j, j = 1, 2, 3$ in Lemma 5 we may write

$$(4.56) \quad u_{k_2} = u_{k_2 1} + u_{k_2 2} + u_{k_2 3},$$

where

$$u_{k_21} = -(\mathbb{E}_{k_2} - \mathbb{E}_{k_2-1})((\beta_{k_2}^{\text{tr}})^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_2}^{-2}(z) \mathbf{s}_{k_2} \eta_{k_2}),$$

and

$$u_{k_22} = (\mathbb{E}_{k_2} - \mathbb{E}_{k_2-1})(\beta_{k_2}(\beta_{k_2}^{\text{tr}})^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_2}^{-2}(z) \mathbf{s}_{k_2} \eta_{k_2}^2), \quad u_{k_23} = (\mathbb{E}_{k_2} - \mathbb{E}_{k_2-1})(\beta_{k_2}^{\text{tr}} \gamma_{k_2}).$$

We claim that

$$(4.57) \quad n^{-3} \sum_{k_1 \neq k_2}^n \mathbb{E}[u_{k_2j} E_{k_2}(u_{k_1}^2)] = O(n^{-2}v^{-2}), \quad j = 1, 2.$$

Indeed, the estimates for q_2, q_3, q_5, q_6 involved in $\mathbb{E}_{k_2}(u_{k_1}^2)$ are straightforward by the argument from (4.48) to (4.52) in Lemma 5 for $\mathbb{E}(u_k^2)$. Here and below, $q_j, j = 1, \dots, 6$ are obtained, respectively, from $\{q_j\}$ in Lemma 5 with k replaced by k_1 . To deal with q_1 , from (4.48) and (4.50) we see that

$$n^{-3} \sum_{k_1 \neq k_2}^n \mathbb{E}[u_{k_22} \mathbb{E}_{k_2}(q_1)] = O(n^{-2}v^{-2}).$$

As for u_{k_21} , we use (3.20) and Lemma 3 first to obtain

$$(4.58) \quad \mathbb{E}_{k-1}((\beta_k^{\text{tr}})^2 \gamma_k \eta_k) = \mathbb{E}_{k-1}((\beta_k^{\text{tr}})^2 n^{-2} \text{tr} \mathbf{A}_k^{-3}(z)) + O(n^{-2}v^{-3}).$$

By (3.20), (3.24), (4.58), (4.49), Lemmas 3, 8, 9 and 2 we then have

$$\begin{aligned} \mathbb{E}[u_{k_21} \mathbb{E}_{k_2}(q_1)] &= n^{-1} \mathbb{E}[u_{k_21} \mathbb{E}_{k_2}(\beta_{k_1}^{\text{tr}} \beta_{k_1}^{\text{tr}} n^{-1} \text{tr} \mathbf{A}_{k_1}^{-2}(z) \underline{\mathbf{A}}_{k_1}^{-2}(z))] + O(n^{-1}v^{-2}) \\ &= n^{-1} \mathbb{E}[u_{k_21} \mathbb{E}_{k_2}(\beta_{k_1 k_2}^{\text{tr}} \beta_{k_1 k_2}^{\text{tr}} n^{-1} \text{tr} \mathbf{A}_{k_1 k_2}^{-2}(z) \underline{\mathbf{A}}_{k_1 k_2}^{-2}(z))] + O(n^{-1}v^{-2}) \\ &= n^{-2} \mathbb{E}[n^{-1} \text{tr} \mathbf{A}_{k_1 k_2}^{-3}(z) (\beta_{k_2}^{\text{tr}})^2 \mathbb{E}_{k_2}(\beta_{k_1 k_2}^{\text{tr}} \beta_{k_1 k_2}^{\text{tr}} n^{-1} \text{tr} \mathbf{A}_{k_1 k_2}^{-2}(z) \underline{\mathbf{A}}_{k_1 k_2}^{-2}(z))] + O(n^{-1}v^{-2}) \\ &= O(n^{-2}v^{-4}), \end{aligned}$$

where we use the fact that via (3.24)

$$(4.59) \quad |n^{-1} \text{tr} \mathbf{A}_{k_1}^{-2}(z) \underline{\mathbf{A}}_{k_1}^{-2}(z) - n^{-1} \text{tr} \mathbf{A}_{k_1 k_2}^{-2}(z) \underline{\mathbf{A}}_{k_1 k_2}^{-2}(z)| \leq M/(nv^4).$$

To handle q_4 , by (4.50), (4.48) and (4.49), it is straightforward to check that

$$n^{-3} \sum_{k_1 \neq k_2}^n \mathbb{E}[u_{k_22} \mathbb{E}_{k_2}(q_4)] = O(n^{-2}v^{-2}).$$

As for u_{k_21} , it follows from (4.49), (4.58), Lemmas 2 and 9 that

$$n^{-1} \mathbb{E}[u_{k_21} \mathbb{E}_{k_2}(q_4)] = n^{-1} b_1^3 \mathbb{E}[u_{k_21} \mathbb{E}_{k_2}(\mathbb{E}_{k_1}(\gamma_{k_1}) \mathbb{E}_{k_1}(\gamma_{k_1} \eta_{k_1}))] + O(n^{-2}v^{-2}) = O(n^{-2}v^{-2}),$$

where we use the fact that the arguments for (4.35) are also applicable to $\mathbb{E}_{k_2}(\mathbb{E}_{k_1}(\gamma_{k_1}) \mathbb{E}_{k_1}(\gamma_{k_1} \eta_{k_1}))$.

Next consider $\mathbb{E}[u_{k_23} \mathbb{E}_{k_2}(u_{k_1}^2)]$. The strategy is to remove \mathbf{s}_{k_2} from $\mathbb{E}_{k_2}(u_{k_1}^2)$ so that we make use of the fact that

$$(4.60) \quad \mathbb{E}[u_{k_23} \mathbb{E}_{k_2}(u_{k_14}^2)] = 0$$

by the fact that $u_{k_1 4}$ is independent of \mathbf{s}_{k_2} with $u_{k_1 4} = (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})(\beta_{k_2 k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1})$. To this end, write

$$u_{k_1} = u_{k_1 1} + u_{k_1 2} + u_{k_1 3} + u_{k_1 4},$$

where

$$\begin{aligned} u_{k_1 1} &= (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})((\beta_{k_1} - \beta_{k_2 k_1}) \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \\ u_{k_1 2} &= (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})[(\beta_{k_1} - \beta_{k_2 k_1}) \mathbf{s}_{k_1}^T (\mathbf{A}_{k_1}^{-2}(z) - \mathbf{A}_{k_1 k_2}^{-2}(z)) \mathbf{s}_{k_1}] \\ u_{k_1 3} &= (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})(\beta_{k_2 k_1} \mathbf{s}_{k_1}^T (\mathbf{A}_{k_1}^{-2}(z) - \mathbf{A}_{k_1 k_2}^{-2}(z)) \mathbf{s}_{k_1}). \end{aligned}$$

We now substitute $u_{k_1 1} + u_{k_1 2} + u_{k_1 3} + u_{k_1 4}$ for u_{k_1} in $\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(u_{k_1}^2)]$ and evaluate them one by one besides using (4.60). By (3.5), (3.7), Lemmas 2 and 8 we have

$$(4.61) \quad \mathbb{E}|\beta_{k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1}^{-2}(z) \mathbf{s}_{k_1}|^8 \leq M \mathbb{E}|\gamma_{k_1}|^8 + M|n^{-1} \mathbb{E} \mathbf{A}_{k_1}^{-2}(z)|^8 + M v^{-8} \mathbb{E}|\xi_{k_1}|^8 + M \mathbb{E}|\Gamma_1^{(2)}|^8 \leq M/v^4,$$

and via (3.25)

$$(4.62) \quad \begin{aligned} \mathbb{E}|\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2}|^2 \beta_{k_2 k_1} \beta_{k_1 k_2}|^4 &\leq M(\mathbb{E}|\zeta_1(z) \beta_{k_1 k_2}|^8 \mathbb{E}|\beta_{k_2 k_1}|^8)^{1/2} \\ &+ M n^{-4} v^{-4} \mathbb{E}|\beta_{k_2 k_1}|^4 \leq M n^{-4} v^{-4}, \end{aligned}$$

where $\zeta_1(z) = \text{center}^{k_1}(\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_1})$. Combining (4.61), (4.62) and (4.48) we obtain

$$\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(u_{k_1 1} + u_{k_1 2})^2] = O(n^{-2} v^{-7/2}).$$

As in (4.62) one may verify that

$$(4.63) \quad \mathbb{E}|\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1} \beta_{k_1 k_2}|^4 \leq M/(n v^2).$$

Thus from (4.61), (4.62), (4.48), (4.63), (3.20), (3.25) and Lemmas 2, 9, 8 we obtain

$$\begin{aligned} &\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(u_{k_1 3})^2] \\ &= 2 \mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})(\beta_{k_2 k_1} \mathbf{s}_{k_1}^T G_{k_1 k_2}(z) \mathbf{s}_{k_1} \beta_{k_1 k_2}))^2] + O(n^{-2} v^{-7/2}) \\ &= 2 b_{12}(z) \mathbb{E}[u_{k_2 3} E_{k_2}(\mathbb{E}_{k_1}((\mathbf{s}_{k_1}^T G_{k_1 k_2}(z) \mathbf{s}_{k_1} - \frac{1}{n} \text{tr} G_{k_1 k_2}(z)) \beta_{k_1 k_2}))^2] + O(n^{-2} v^{-7/2}) \\ &= 2 b_{12}(z) n^{-2} \mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(\beta_{k_1 k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_2} \beta_{k_1 k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2})] \\ &\quad + O(n^{-2} v^{-7/2}) = O(n^{-2} v^{-4}), \end{aligned}$$

where $G_{k_1 k_2}(z) = \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z)$. In view of (4.61), (4.62) and (4.48) we also conclude that

$$\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}((u_{k_1 1} + u_{k_1 2}) u_{k_1 3})] = O(n^{-1} v^{-2}),$$

because via (4.63), (4.62) and (4.62)

$$(4.64) \quad \mathbb{E}|\beta_{k_2 k_1} \mathbf{s}_{k_1}^T (\mathbf{A}_{k_1}^{-2}(z) - \mathbf{A}_{k_1 k_2}^{-2}(z)) \mathbf{s}_{k_1}|^2 \leq M.$$

Likewise, by (4.61), (4.62), (4.48) and (4.64) we have

$$\mathbb{E}[u_{k_2 3} E_{k_2}(u_{k_1 2} u_{k_1 4})] = O(n^{-1} v^{-2}).$$

Moreover by (3.5), (4.61), (4.62), (4.48), (3.20) and (3.7), we have

$$\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(u_{k_1 1} u_{k_1 4})]$$

$$\begin{aligned}
&= b_1 \mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_2 k_1} \rho_{k_1 k_2} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) u_{k_1 4})] + O(n^{-1} v^{-2}) \\
&= n^{-2} b_1 \mathbb{E} [(\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_2 k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbb{E}_{k_2} (\beta_{k_2}^{\text{tr}} \mathbf{A}_{k_1 k_2}^{-1}(z)) \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \\
&\quad \times (u_{k_1 4})] + O(n^{-2} v^{-4}) = O(n^{-2} v^{-4}),
\end{aligned}$$

where

$$\rho_{k_1 k_2} = \text{center}^{k_2} (\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2})$$

and we also use the fact that

$$(4.65) \quad \mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_2 k_1} n^{-1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) u_{k_1 4})] = 0.$$

As for $\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} (u_{k_1 3} u_{k_1 4})]$, on the one hand, using (3.20) twice and Lemma 8 we obtain

$$\begin{aligned}
&\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (b_{12} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) u_{k_1 4})] \\
&= \mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (b_{12} \rho_{k_1 k_2}^{(2)}) u_{k_1 4})] \\
&= n^{-2} b_{12}^2 \mathbb{E} [(\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbb{E}_{k_2} (\mathbf{A}_{k_1 k_2}^{-2}) \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_1}) u_{k_1 4}] + O(n^{-2} v^{-4}) \\
&= n^{-2} b_{12}^3 \mathbb{E} [(\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbb{E}_{k_2} (\mathbf{A}_{k_1 k_2}^{-2}) \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_1}) \mathbb{E}_{k_1} (\gamma_{k_2 k_1})] + O(n^{-2} v^{-4}) \\
&= n^{-3} b_{12}^3 \mathbb{E} [n^{-1} \text{tr} \mathbf{A}_{k_1 k_2}^{-1} \mathbb{E}_{k_2} (\mathbf{A}_{k_1 k_2}^{-2}) \mathbf{A}_{k_1 k_2}^{-2} \mathbb{E}_{k_1} (\mathbf{A}_{k_1 k_2}^{-2})] + O(n^{-2} v^{-4}) = O(n^{-2} v^{-4}),
\end{aligned}$$

where we also use an equality similar to (4.65),

$$\rho_{k_1 k_2}^{(2)} = \text{center}^{k_2} (\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2})$$

and

$$\gamma_{k_2 k_1} = \text{center}^{k_1} (\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}).$$

Similarly one may verify that

$$\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_1 k_2} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \mathbb{E}_{k_1} (\gamma_{k_2 k_1} b_{12}))] = O(n^{-2} v^{-4}).$$

Moreover by Hölder's inequality, (4.48), (4.63) and Lemma 8

$$\begin{aligned}
&\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (b_{12} \beta_{k_1 k_2} \xi_{k_1 k_2} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \\
&\quad \times (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (b_{12} \beta_{k_1 k_2} \xi_{k_1 k_2}))] = O(n^{-2} v^{-4}).
\end{aligned}$$

Via (3.26), these ensure that

$$\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_1 k_2} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) u_{k_1 4})] = O(n^{-2} v^{-4}).$$

On the other hand, apparently from Hölder's inequality, (4.61), (4.62), (4.63), (4.48) and (3.15) we obtain

$$\begin{aligned}
&\mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} ((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_1 k_2}^2 (\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2})^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \\
&\quad \times (\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1}) (\beta_{k_2 k_1} \beta_{k_2 k_1}^{\text{tr}} \eta_{k_2 k_1} \mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}))] = O(n^{-2} v^{-4}).
\end{aligned}$$

Similar to (4.36) we obtain

$$\begin{aligned}
(4.66) \quad &\mathbb{E}_{k_2} [\mathbb{E}_{k_1} (\rho_{k_2 k_1} \beta_{k_1 k_2}^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \gamma_{k_2 k_1})] \\
&= n^{-5/2} [\mathbb{E} [(X_{11}^2 - 1)^2 X_{11}] \sum_{i=1}^p \mathbb{E}_{k_2} ((\mathbf{B}_1)_{ii} (\mathbf{B}_2)_{ii} y_i)]
\end{aligned}$$

$$\begin{aligned}
& +2\mathbb{E}X_{11}^3 \sum_{i_1 \neq j_1} \mathbb{E}_{k_2} ((\mathbf{B}_1)_{i_1 j_1} [(\mathbf{B}_2)_{j_1 j_1} y_{i_1} + (\mathbf{B}_2)_{i_1 i_1} y_{j_1}]) \\
& +2\mathbb{E}X_{11}^3 \sum_{i_1 \neq j_1} \mathbb{E}_{k_2} ((\mathbf{B}_1)_{i_1 j_1} (\mathbf{B}_2)_{i_1 j_1} (y_{i_1} + y_{j_1}))),
\end{aligned}$$

where

$$\mathbf{B}_1 = \beta_{k_1 k_2}^2 \mathbf{A}_{k_1 k_2}^{-1} \mathbf{s}_{k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1}, \quad \mathbf{B}_2 = \mathbf{A}_{k_1 k_2}^{-2}, \quad y_i = \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2} \mathbf{e}_i).$$

We obtain from Lemma 8, Lemma 3 and Burkholder's inequality

$$\mathbb{E} |\beta_{k_1 k_2} (\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_i)^2|^4 \leq M (\mathbb{E} |\beta_{k_1 k_2}|^8 \mathbb{E} |\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_i|^{16})^{1/2} \leq M/(n^4 v^7).$$

Similarly

$$(4.67) \quad \mathbb{E} |\beta_{k_1 k_2} y_i|^4 \leq M (\mathbb{E} |\beta_{k_1 k_2}|^8 \mathbb{E} |\beta_{k_1 k_2}^{\text{tr}} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2} \mathbf{e}_i|^8)^{1/2} \leq M/(n v^{11/2}).$$

These, Lemma 3 and (4.48) imply that

$$\mathbb{E} [u_{k_2 3} n^{-5/2} \sum_{i=1}^p \mathbb{E}_{k_2} ((\mathbf{B}_1)_{ii} (\mathbf{B}_2)_{ii} y_i)] = O(n^{-2} v^{-4}).$$

Note that

$$\begin{aligned}
n^{-5/2} \sum_{i_1 \neq j_1} (\mathbf{B}_1)_{ij} (\mathbf{B}_2)_{j_1 j_1} y_{i_1} &= n^{-5/2} \beta_{k_1 k_2}^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_2}) \\
&\times \sum_j \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_j (\mathbf{B}_2)_{jj} + n^{-5/2} \sum_{i_1 \neq j_1} (\mathbf{B}_1)_{ij} (\mathbf{B}_2)_{j_1 j_1} y_{i_1}.
\end{aligned}$$

By Lemma 8, Lemma 3 and Burkholder's inequality we also have

$$(4.68) \quad \mathbb{E} |\beta_{k_1 k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_i|^4 \leq M (\mathbb{E} |\beta_{k_1 k_2}|^8 \mathbb{E} |\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_i|^8)^{1/2} \leq M/(n^4 v^6)$$

and via (3.25), (3.6)

$$\begin{aligned}
\mathbb{E} |\beta_{k_1 k_2} \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} E_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_2})|^8 &\leq M v^{-4} \mathbb{E} (|\sqrt{|\beta_{k_1 k_2}|} \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_2})|^8) \\
&\leq M v^{-4} (\mathbb{E} |\beta_{k_1 k_2}|^8 \mathbb{E} |(\beta_{k_1 k_2}^{\text{tr}})^2 \mathbf{s}_{k_2} \mathbf{A}_{k_1 k_2}^{-2} (z) \mathbf{A}_{k_1 k_2}^{-2} (\bar{z}) \mathbf{s}_{k_2}|^8)^{1/2} \leq M/v^{16}
\end{aligned}$$

These, together with (4.48), Lemmas 8 and 3, ensure that

$$\mathbb{E} [u_{k_2 3} n^{-5/2} \sum_{i_1 \neq j_1} (\mathbf{B}_1)_{ij} (\mathbf{B}_2)_{j_1 j_1} y_{i_1}] = O(n^{-2} v^{-4}).$$

This argument also works for the remaining terms in (4.66) so that

$$\mathbb{E} [u_{k_2 3} \times (4.66)] = O(n^{-2} v^{-4}).$$

As in (3.20), a direct calculation, together with (4.68), (4.48) and an estimate similar to (4.67), yields

$$\begin{aligned}
& \mathbb{E} [u_{k_2 3} \mathbb{E}_{k_2} (\mathbb{E}_{k_1-1} (\rho_{k_2 k_1} \beta_{k_1 k_2}^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2} (z) \mathbf{s}_{k_1}) \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \gamma_{k_1 k_2})))] \\
&= n^{-3/2} \mathbb{E} (X_{11}^3 - X_{11}) \sum_{i=1}^p \mathbb{E} [u_{k_2 3} E_{k_2} (\mathbb{E}_{k_1-1} (\beta_{k_1 k_2}^2 (\mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-1} \mathbf{e}_i)^2 \mathbf{e}_i^T \mathbf{A}_{k_1 k_2}^{-2} \mathbf{s}_{k_2}) \mathbb{E}_{k_1} (\beta_{k_1 k_2}^{\text{tr}} \gamma_{k_1 k_2})))] \\
&= O(n^{-2} v^{-4}).
\end{aligned}$$

These ensure

$$\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}((\mathbb{E}_{k_1} - \mathbb{E}_{k_1-1})(\beta_{k_1 k_2}^2 (\mathbf{s}_{k_1}^T \mathbf{A}_{k_1 k_2}^{-1}(z) \mathbf{s}_{k_2})^2 \mathbf{s}_{k_2}^T \mathbf{A}_{k_1 k_2}^{-2}(z) \mathbf{s}_{k_1}) u_{k_1 4})] = O(n^{-2} v^{-4}).$$

Hence

$$\mathbb{E}[u_{k_2 3} \mathbb{E}_{k_2}(u_{k_1 3} u_{k_1 4})] = O(n^{-2} v^{-4}).$$

Thus the proof is completed. \square

5. THE LIMIT OF MEAN FUNCTION

The aim in the section is to find the limit of

$$\frac{1}{2\pi i} \oint K\left(\frac{x-z}{h}\right) n(\mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z)) dz.$$

It is thus sufficient to investigate the uniform convergence $nh(\mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z))$ on the contour. In order to establish Theorem 1 we instead apply the estimates in the last section to investigate $n(\mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z))$.

Write

$$\mathbf{A} - z\mathbf{I} = \sum_{j=1}^n \mathbf{s}_j \mathbf{s}_j^T - z\mathbf{I}.$$

Multiplying both sides by $\mathbf{A}^{-1}(z)$, taking the trace and dividing by n we obtain

$$(5.1) \quad c_n + z c_n m_n(z) = 1 - \frac{1}{n} \sum_{j=1}^n \beta_j(z)$$

(one may see the equality above (2.2) of [21]). Taking expectation on both sides of the equality above and applying (3.5) we have

$$(5.2) \quad c_n + z c_n \mathbb{E} m_n(z) = 1 - b(z) + b(z) D_n,$$

where

$$D_n = \mathbb{E}[\beta_1(z)(\mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1 - \mathbb{E} n^{-1} \text{tr} \mathbf{A}^{-1}(z))].$$

On the other hand, it follows from (2.2) and Lemma 7 that

$$(5.3) \quad c_n + z c_n m_n^0(z) = 1 - (1 + c_n m_n^0(z))^{-1}.$$

Taking the difference between (5.2) and (5.3), along with (5.3), yields

$$(5.4) \quad n(\mathbb{E} m_n(z) - m_n^0(z)) = n D_n / (c_n g(z)),$$

where $g(z)$ is defined in Lemma 1.

Considered $z \in \gamma_1 \cup \gamma_2$ first. Applying (3.5) and (3.4) yields

$$\begin{aligned} & \mathbb{E}(\text{tr} \mathbf{A}_1^{-1}(z)) - \mathbb{E}(\text{tr} \mathbf{A}^{-1}(z)) = \mathbb{E}(\beta_1 \mathbf{s}_1^T \mathbf{A}_1^{-2}(z) \mathbf{s}_1) \\ &= b_1 \mathbb{E}([1 - b_1 \xi_1 + b_1 \beta_1 \xi_1^2(z)] \mathbf{s}_1^T \mathbf{A}_1^{-2}(z) \mathbf{s}_1) \\ (5.5) \quad &= b_1 \mathbb{E} n^{-1} \text{tr} \mathbf{A}_1^{-2}(z) - d_{n1} + d_{n2} + d_{n3}, \end{aligned}$$

where

$$d_{n1} = b_1^2 \mathbb{E}[\eta_1 \eta_1^{(2)}], \quad d_{n2} = b_1^2 \mathbb{E}[\Gamma_1 \mathbf{s}_1^T \mathbf{A}_1^{-2}(z) \mathbf{s}_1] = b_1^2 \mathbb{E}[\Gamma_1 \Gamma_1^{(2)}], \quad d_{n3} = b_1 \mathbb{E}[\beta_1 \xi_1^2 \mathbf{s}_1^T \mathbf{A}_1^{-2} \mathbf{s}_1].$$

It follows from (3.25) and Lemma 8 that

$$|d_{nj}| \leq M/(nv^2), \quad j = 1, 2, 3,$$

which implies

$$(5.6) \quad \mathbb{E}(\text{tr} \mathbf{A}_1^{-1}(z)) - \mathbb{E}(\text{tr} \mathbf{A}^{-1}(z)) = b_1 \mathbb{E} n^{-1} \text{tr} \mathbf{A}_1^{-2}(z) + O(n^{-1}v^{-2}).$$

Next by (3.5)

$$(5.7) \quad \begin{aligned} & n \mathbb{E}[\beta_1 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1] - \mathbb{E}(\beta_1) \mathbb{E}(\text{tr} \mathbf{A}_1^{-1}(z)) \\ &= -nb_1^2 \mathbb{E}[\xi_1 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1] + nb_1^2 \mathbb{E}[\beta_1 \xi_1^2 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1] - b_1^2 \mathbb{E}(\beta_1 \xi_1^2) \mathbb{E}[\text{tr} \mathbf{A}_1^{-1}(z)] \\ &= f_{n1} + f_{n2} + f_{n3} + f_{n4}, \end{aligned}$$

where

$$f_{n1} = -nb_1^2 \mathbb{E} \eta_1^2, \quad f_{n2} = -nb_1^2 \mathbb{E}(\Gamma_1 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1) = nb_1^2 \mathbb{E}(\Gamma_1)^2, \quad f_{n3} = nb_1^2 \mathbb{E}(\beta_1 \xi_1^2 \eta_1),$$

and

$$f_{n4} = b_1^2 [\mathbb{E}(\beta_1 \xi_1^2 \text{tr} \mathbf{A}_1^{-1}) - \mathbb{E}(\beta_1 \xi_1^2) \mathbb{E} \text{tr} \mathbf{A}_1^{-1}].$$

By (3.20) we have

$$(5.8) \quad f_{n1} = -nb_1^2 \mathbb{E} \eta_1^2 = -2b_1^2 \mathbb{E} n^{-1} \text{tr} \mathbf{A}_1^{-2} + O(n^{-1}v^{-2}).$$

By Lemma 5 and (4.9)

$$(5.9) \quad f_{n2}/g(z) = O(n^{-1}v^{-2}|z + c_n - 1 + 2zc_n m_n^0(z)|^{-1}),$$

where we use the fact that via (4.4) and (4.5)

$$(5.10) \quad |g(z)| \geq M_2|z + c_n - 1 + 2zc_n m_n^0(z)|, \quad M_2 > 0.$$

Consider f_{n4} next. Apply (3.5) to further write f_{n4} as

$$f_{n4} = f_{n41} + f_{n42} + f_{n43},$$

where

$$f_{n41} = nb_1^3 \mathbb{E}(\eta_1^2 \Gamma_1), \quad f_{n42} = nb_1^3 \mathbb{E}(\Gamma_1)^3,$$

and

$$f_{n43} = -b_1^3 [\mathbb{E}(\beta_1 \xi_1^3 \text{tr} \mathbf{A}_1^{-1}(z)) - \mathbb{E}(\beta_1 \xi_1^3) \mathbb{E} \text{tr} \mathbf{A}_1^{-1}(z)].$$

By Hölder's inequality and Lemma 8 we obtain

$$|f_{n43}| \leq nM(\mathbb{E}|\beta_1|^2 \mathbb{E}|\xi_1^3 \Gamma_1|^2)^{1/2} \leq M/(nv^2).$$

From (3.20), Lemmas 3 and 8 we conclude that

$$f_{n41} = 2b_1^3 \mathbb{E}(\Gamma_1 \Gamma_1^{(2)}) + O(n^{-1}v^{-2}) = O(n^{-1}v^{-2}).$$

In view of Lemma 6 we also have $f_{n42} = O(n^{-1}v^{-2})$. Therefore $f_{n4} = O(n^{-1}v^{-2})$.

By (3.5) f_{n3} may be further written as

$$f_{n3} = f_{n31} + f_{n32} + f_{n33},$$

where

$$f_{n31} = nb_1^3 \mathbb{E}(\eta_1^3), \quad f_{n32} = 2nb_1^3 \mathbb{E}(\eta_1^2 \Gamma_1), \quad f_{n33} = nb_1^3 \mathbb{E}(\beta_1 \xi_1^3 \eta_1).$$

Note that $f_{n32} = 2f_{n41}$. Lemmas 8 and 4 ensure, respectively, $f_{n33} = O(n^{-1}v^{-2})$ and $f_{n31} = O(n^{-1}v^{-2})$. We then conclude that $f_{n3} = O(n^{-1}v^{-2})$.

Summarizing the above argument (particularly (5.4), (5.6) and (5.8)) we obtain

$$(5.11) \quad nD_n/g(z) = -(b_1^2/g(z))\mathbb{E}n^{-1}\mathrm{tr}\mathbf{A}_1^{-2}(z) + O(n^{-1}v^{-2}|z + c_n - 1 + 2zc_n m_n^0(z)|^{-1}).$$

We would point out that (5.11), Lemmas 1 and 2 imply

Proposition 1. *For $v > M/\sqrt{n}$ and $u \in [a, b]$,*

$$(5.12) \quad |\mathbb{E}m_n(z) - m(z)| \leq M/(nv).$$

From (4.11) we have

$$(5.13) \quad n^{-1}\mathbb{E}[\mathrm{tr}\mathbf{A}_1^{-2}(z)] = \frac{c_n}{z^2(1 + \underline{m}_n^0(z))^2} \left(1 - \frac{c_n(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2}\right)^{-1} + O(n^{-1}v^{-2}).$$

We then conclude from (5.4), (5.12), (5.11), (4.12), (3.57) and (5.13) that

$$(5.14) \quad n(\mathbb{E}\underline{m}_n(z) - \underline{m}_n^0(z)) = \frac{c_n(\underline{m}_n^0(z))^3}{(1 + \underline{m}_n^0(z))^3} \left(1 - \frac{c_n(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2}\right)^{-2} + O(n^{-1}v^{-2}|z + c_n - 1 + 2zc_n m_n^0(z)|^{-1}).$$

The case where z lies in the vertical lines on the contour can be handled similarly as pointed out in the last section with the truncation version of $X_n(z)$.

In view of (3.45) it remains to find the limit of the following

$$(5.15) \quad \frac{1}{4\pi i} \oint K\left(\frac{x-z}{h}\right) \frac{c_n(\underline{m}_n^0(z))^3}{(1 + \underline{m}_n^0(z))^3} \left(1 - \frac{c_n(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2}\right)^{-2} dz,$$

which is done in Appendix 3.

6. THE PROOF OF THEOREMS 3 AND 1

Proof of Theorem 3. Let $x \in (a, b)$. We claim that

$$\begin{aligned} & nh \left[h^{-1} \int_a^b K\left(\frac{x-y}{h}\right) d\mathbb{F}_{c_n}(y) - f_{c_n}(x) \right] \\ &= nh \left[\int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K(y) f_{c_n}(x-yh) dy - f_{c_n}(x) \right] \\ &= nh \left[f_{c_n}(x) \int_{\frac{x-b}{2h}}^{\frac{x-a}{2h}} K(y) dy - f_{c_n}(x) \right] + \text{remainder}, \end{aligned}$$

where

$$|\text{remainder}| \leq 4nh^3((x-a)^{-2} + (b-x)^{-2})(\|f\| + M) \int y^2 |K(y)| dy \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Indeed, by Taylor's expansion

$$f_{c_n}(x-yh) = f_{c_n}(x) - f'_{c_n}(x)yh + f''_{c_n}(x-\theta yh)(yh)^2,$$

where $\theta \in [0, 1]$. Moreover note that

$$nh^2 \left| \int_{\frac{x-a}{2h}}^{+\infty} yK(y)dy + \int_{-\infty}^{\frac{x-b}{2h}} yK(y)dy \right| \leq 4nh^3((x-a)^{-2} + (b-x)^{-2}) \int y^2|K(y)|dy \rightarrow 0,$$

$$nh \left[1 - \int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K(y)dy \right] \leq 4nh^3((x-a)^{-2} + (b-x)^{-2}) \int y^2|K(y)|dy \rightarrow 0,$$

and $f''_{c_n}(x - \theta y h)$ is bounded above by a finite constant depending only on x when $y \in ((x-b)/(2h), (x-a)/(2h))$. Thus the proof is complete. \square

Proof of Theorem 1. Following the truncation steps in [3] we could truncate and re-normalize the random variables so that

$$(6.1) \quad |X_{ij}| \leq \tau_n n^{1/2}, \quad \mathbb{E}X_{ij} = 0, \quad \mathbb{E}X_{ij}^2 = 1,$$

where $\tau_n n^{1/3} \rightarrow \infty$ and $\tau_n \rightarrow 0$. Based on this one may then verify that

$$(6.2) \quad \mathbb{E}X_{11}^4 = 3 + O\left(\frac{1}{n}\right).$$

For any finite constants l_1, \dots, l_d , by Cauchy's theorem and Fubini's theorem we write

$$\begin{aligned} (6.3) \quad & \frac{n}{\sqrt{\ln h^{-1}}} \sum_{j=1}^d l_j \left(F_n(x_j) - \int_{-\infty}^{x_j} \frac{1}{h} \int K\left(\frac{t-y}{h}\right) d\mathbb{F}_{c_n}(y) dt \right) \\ &= \frac{n}{\sqrt{\ln h^{-1}}} \sum_{j=1}^d l_j \left(\int_{-\infty}^{x_j} f_n(t) dt - \int_{-\infty}^{x_j} \frac{1}{h} \int K\left(\frac{t-y}{h}\right) d\mathbb{F}_{c_n}(y) dt \right) \\ &= -\frac{n}{2h\pi i \sqrt{\ln h^{-1}}} \sum_{j=1}^d l_j \left(\int_{-\infty}^{x_j} \oint_{\mathcal{C}_1} K\left(\frac{t-z}{h}\right) (\text{tr} \mathbf{A}^{-1}(z) - nm_n^0(z)) dz dt \right) \\ &= -\frac{n}{2h\pi i \sqrt{\ln h^{-1}}} \sum_{j=1}^d l_j \oint_{\mathcal{C}_1} \int_{-\infty}^{x_j} K\left(\frac{t-z}{h}\right) dt (\text{tr} \mathbf{A}^{-1}(z) - nm_n^0(z)) dz, \end{aligned}$$

where the contour \mathcal{C}_1 is defined as before.

Furthermore, we conclude from (3.9) and integration by parts that

$$\begin{aligned} & \frac{1}{2h\pi i \sqrt{\ln h^{-1}}} \oint_{\mathcal{C}_1} \int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt (\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)) dz \\ &= -\frac{1}{2h\pi i \sqrt{\ln h^{-1}}} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint_{\mathcal{C}_1} \int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt [\ln \beta_k(z)]' dz \\ (6.4) \quad &= \frac{1}{2h\pi i \sqrt{\ln h^{-1}}} \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) \oint_{\mathcal{C}_1} K\left(\frac{x-z}{h}\right) \ln \left(\frac{\beta_k^{\text{tr}}(z)}{\beta_k(z)} \right) dz, \end{aligned}$$

where in the last step one uses the fact that via (2.6)

$$(6.5) \quad \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right]' = K\left(\frac{x-z}{h}\right).$$

It is observed that the unique difference between (6.4) and (3.10) is that the test function $K'(\frac{x-z}{h})$ there is replaced by $K(\frac{x-z}{h})$. Therefore, repeating the arguments in Section 2 we obtain that (6.4) is asymptotically normal with covariance (see (3.64) and (3.70))

$$(6.6) \quad -\frac{1}{2h^2\pi^2 \ln h^{-1}} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K\left(\frac{x_1-z_1}{h}\right) K\left(\frac{x_2-z_2}{h}\right) \ln(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) dz_1 dz_2 + O\left(\frac{1}{nh^2}\right).$$

Also, for the nonrandom part we have

$$(6.7) \quad \frac{1}{2h\pi i \sqrt{\ln h^{-1}}} \oint_{\mathcal{C}_1} \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] n(\mathbb{E} \text{tr} \mathbf{A}^{-1}(z) - nm_n^0(z)) dz.$$

Note that

$$|h^{-1} \int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt| < \infty.$$

Likewise, repeating the arguments in Section 3 we see that (6.7) becomes (see (5.14))

$$(6.8) \quad \frac{1}{4h\pi i \sqrt{\ln h^{-1}}} \oint \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] \frac{c_n(\underline{m}_n^0(z))^3}{(1 + \underline{m}_n^0(z))^3} \left(1 - \frac{c_n(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2}\right)^{-2} dz + O\left(\frac{1}{nh^2 \sqrt{\ln h^{-1}}}\right).$$

The limits of (6.6) and (6.8) are derived in Appendix 3.

Applying a change of variables and Fubini's theorem we obtain

$$(6.9) \quad \begin{aligned} \int_{-\infty}^x \left[\frac{1}{h} \int K\left(\frac{t-y}{h}\right) d\mathbb{F}_{c_n}(y) \right] dt &= \int_{-\infty}^x \left(\int K(y) f_{c_n}(t-hy) dy \right) dt \\ &= \int \left(K(y) \int_{-\infty}^x f_{c_n}(t-hy) dt \right) dy = \int K(y) \mathbb{F}_{c_n}(x-hy) dy. \end{aligned}$$

By Taylor's expansion we have

$$\mathbb{F}_{c_n}(x-hy) = \mathbb{F}_{c_n}(x) + hy f_{c_n}(x) + 2^{-1} h^2 y^2 f'_{c_n}(x - \theta hy),$$

where $\theta \in (0, 1)$. This, together with (6.9), yields that

$$\begin{aligned} & \int_{-\infty}^x \left[\frac{1}{h} \int K\left(\frac{t-y}{h}\right) d\mathbb{F}_{c_n}(y) \right] dt \\ &= \int_{|y| \leq x_0/(2h)} K(y) (\mathbb{F}_{c_n}(x) + hy f_{c_n}(x) + 2^{-1} h^2 y^2 f'_{c_n}(x - \theta hy)) dy \\ & \quad + \int_{|y| > x_0/(2h)} K(y) \mathbb{F}_{c_n}(x-hy) dy, \end{aligned}$$

where $x_0 = \min(x, x - a, b - x)$ is positive since $x \in (a, b)$. Note that

$$\begin{aligned} \left| \int_{|y| > x_0/(2h)} K(y) \mathbb{F}_{c_n}(x - hy) dy \right| &\leq (4h^2/x_0^2) \int y^2 |K(y)| dy, \\ \left| \int_{|y| \leq x_0/(2h)} y K(y) dy \right| &= \left| \int_{|y| > x_0/(2h)} y K(y) dy \right| \leq (2h/x_0) \int y^2 |K(y)| dy \end{aligned}$$

and $f'_{c_n}(x - \theta hy)$ is bounded above by a finite constant depending only on x , and that

$$\begin{aligned} \frac{n}{\sqrt{\ln h^{-1}}} \left(F_n(x) - \int_{-\infty}^x \left[\frac{1}{h} \int K\left(\frac{t-y}{h}\right) d\mathbb{F}_{c_n}(y) \right] dt \right) \\ = \frac{n}{\sqrt{\ln h^{-1}}} \left(F_n(x) - \mathbb{F}_{c_n}(x) \right) + O\left(\frac{nh^2}{2\sqrt{\ln h^{-1}}}\right). \end{aligned}$$

Hence the proof of Theorem 1 is complete. \square

7. THE PROOF OF THEOREM 2

For any x , write

$$\mathbb{P}\left(\frac{n}{\sqrt{\ln n}}(x_{n,\alpha} - x_\alpha) \leq x\right) = \mathbb{P}\left(F_n\left(x_\alpha + \frac{x\sqrt{\ln n}}{n}\right) \geq \alpha\right) = \mathbb{P}(\hat{F}_n(x) \geq g_n(x)),$$

where

$$\hat{F}_n(x) = \frac{n}{\sqrt{\ln n}} \left[F_n\left(x_\alpha + \frac{x\sqrt{\ln n}}{n}\right) - \mathbb{F}_{c_n}\left(x_\alpha + \frac{x\sqrt{\ln n}}{n}\right) \right]$$

and

$$g_n(x) = \frac{n}{\sqrt{\ln n}} \left[\alpha - \mathbb{F}_{c_n}\left(x_\alpha + \frac{x\sqrt{\ln n}}{n}\right) \right].$$

By Taylor's expansion we have

$$g_n(x) \rightarrow -x f_c(x_\alpha)$$

and

$$\hat{F}_n(x) = \frac{n}{\sqrt{\ln n}} [F_n(x_\alpha) - \mathbb{F}_{c_n}(x_\alpha)] + o_p(1),$$

where we use Theorem 3 and the fact that $F_n(x)$ and $\mathbb{F}_c(x)$ are both continuous. Theorem 2 then follows the above and Theorem 1.

8. APPENDIX 1

This Appendix collects some frequently used Lemmas.

Lemma 7. (Lemma 2.2 of [3]) Suppose that X_1, \dots, X_n are i.i.d real random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Let $\mathbf{x} = (X_1, \dots, X_n)^T$ and \mathbf{D} be any $n \times n$ complex matrix. Then for any $p \geq 2$

$$\mathbb{E}|\mathbf{x}^T \mathbf{D} \mathbf{x} - \text{tr} \mathbf{D}|^p \leq M_p \left[(\mathbb{E}|X_1|^4 \text{tr} \mathbf{D} \mathbf{D}^*)^{p/2} + \mathbb{E}|X_1|^{2p} \text{tr}(\mathbf{D} \mathbf{D}^*)^{p/2} \right].$$

Lemma 8. *Assume that $v \geq M/\sqrt{n}$ and $u \in [a, b]$. Then*

$$|\underline{m}_n^0(z)| \leq M, \quad |\mathbb{E}m_n(z)| \leq M, \quad |b_1(z)| \leq M, \quad \mathbb{E}|\beta_k^{\text{tr}}(z)|^8 \leq M, \quad \mathbb{E}|\beta_k(z)|^8 \leq M;$$

$$(8.1) \quad \mathbb{E}|\Gamma_k|^8 \leq M/(n^8 v^{12}), \quad \mathbb{E}|\Gamma_k^{(2)}|^8 \leq M/(n^8 v^{20}), \quad \mathbb{E}|\xi_k(z)|^8 \leq M/(n^4 v^4);$$

for $m_1 = 1, 2$ and $m_2 = 0, 1, 2, m_3 = 0, 1, 2$,

$$(8.2) \quad \mathbb{E}|\text{center}^k(\mathbf{s}_k^T \mathbf{A}_k^{-m_1}(z) \underline{\mathbf{A}}_k^{-m_2}(z) \mathbf{A}_k^{-m_3}(z) \mathbf{s}_k)|^8 \leq M/(n^4 v^{8m_1+8m_2+8m_3-4}),$$

where center^k is defined in (3.31) and $\underline{\mathbf{A}}_k^{-1}(z)$ defined right before Section 3.1;

$$(8.3) \quad |u_n(z)| = |z - (1 - n^{-1})b_{12}(z)|^{-1} \leq M.$$

Remark 4. *Checking the argument of Lemma 8 indicates that all above estimates involving $\mathbf{A}_k^{-1}(z)$ (and $\underline{\mathbf{A}}_k^{-1}(z)$) still hold if replacing $\mathbf{A}_k^{-1}(z)$ (and $\underline{\mathbf{A}}_k^{-1}(z)$) by $\mathbf{A}_{kj}^{-1}(z)$ (and $\underline{\mathbf{A}}_{kj}^{-1}(z)$) respectively.*

Proof. As pointed out in (6.1) in [11], we obtain

$$(8.4) \quad |\underline{m}_n^0(z)| \leq M, \quad |m_n^0(z)| \leq M.$$

It was proved in [10] that

$$(8.5) \quad |\mathbb{E}m_n(z)| \leq M, \quad |\mathbb{E}\underline{m}_n(z)| \leq M, \quad |b_1(z)| \leq M.$$

See Lemma 6.2 of [10] for the first estimate of (8.1) and Cauchy's theorem ensures the second estimate of (8.1) via the first estimate of (8.1).

Write

$$(8.6) \quad \beta_1^{\text{tr}} = b_1 - \beta_1^{\text{tr}} b_1 \Gamma_1 = b_1 - b_1^2 \Gamma_1 + \beta_1^{\text{tr}} b_1^2 \Gamma_1^2.$$

We then conclude from (3.8), (8.6) and Lemma 6.2 of [10] that

$$(8.7) \quad \mathbb{E}|\beta_1^{\text{tr}}|^8 \leq M(1 + v^{-8} E|\Gamma_1|^{16}) \leq M.$$

Expand $\beta_1(z)$ as

$$(8.8) \quad \beta_1 = \beta_1^{\text{tr}} - \beta_1^{\text{tr}} \beta_1 \eta_1 = \beta_1^{\text{tr}} - (\beta_1^{\text{tr}})^2 \eta_1 + (\beta_1^{\text{tr}})^2 \beta_1 \eta_1^2.$$

It follows from (8.7), (3.6) and Lemma 7 that

$$\mathbb{E}|\beta_1(z)|^8 \leq M + ME|(\beta_1^{\text{tr}})^2 \eta_1|^8 + Mv^{-8} \mathbb{E}|\eta_1(z) \beta_1^{\text{tr}}(z)|^{16} \leq M.$$

From (3.24) and (8.5) we have

$$(8.9) \quad |n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z)| \leq M.$$

As for (8.2) by Lemma 7, (3.6), (8.1) and (8.9) we then obtain

$$\begin{aligned} & \mathbb{E}|\text{center}^k(\mathbf{s}_k^T \mathbf{A}_k^{-m_1}(z) \underline{\mathbf{A}}_k^{-m_2}(z) \mathbf{A}_k^{-m_3}(z) \mathbf{s}_k)|^8 \\ & \leq Mn^{-4} \mathbb{E}(n^{-1} \text{tr} \mathbf{A}_k^{-m_1}(z) \underline{\mathbf{A}}_k^{-m_2}(z) \mathbf{A}_k^{-m_3}(z) \mathbf{A}_k^{-m_3}(\bar{z}) \underline{\mathbf{A}}_k^{-m_2}(\bar{z}) \mathbf{A}_k^{-m_1}(z))^4 \\ & \leq Mn^{-4} v^{-8m_1-8m_2-8m_3+4} (\mathbb{E}|\Gamma_k|^4 + |\Im(n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z))|^4) \leq Mn^{-4} v^{-8m_1-8m_2-8m_3+4}, \end{aligned}$$

where $\mathbf{A}_k^{-1}(\bar{z})$ denotes the complex conjugate of $\mathbf{A}_k^{-1}(z)$ and we also use the fact that

$$n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{A}_k^{-1}(\bar{z}) = v^{-1} \Im(n^{-1} \mathbb{E} \text{tr} \mathbf{A}_k^{-1}(z)).$$

This, together with (8.1), yields the estimate of $\xi_1(z)$.

Via (2.4) we have

$$(8.10) \quad |z + z\underline{m}_n^0(z)|^{-1} = |z + z c_n m_n^0(z) - 1 + c_n|^{-1} = |m_n^0(z)| \leq M$$

which implies that $(z + (1 - n^{-1})z\underline{m}_n^0(z))^{-1}$ is bounded. By (8.10), (3.57) and the equality

$$u_n(z) - (z + (1 - n^{-1})z\underline{m}_n^0(z))^{-1} = (1 - n^{-1})u_n(z)(b_{12}(z) + z\underline{m}_n^0(z))(z + (1 - n^{-1})z\underline{m}_n^0(z))^{-1}$$

we obtain

$$|u_n(z)| \leq M(1 - n^{-1}v^{-3/2})^{-1}|z + (1 - n^{-1})z\underline{m}_n^0(z)|^{-1} \leq M.$$

This implies (8.3). \square

Lemma 9. Assume that $v \geq M_3/\sqrt{n}$ with M_3 being sufficiently large and $u \in [a, b]$. Then

$$(8.11) \quad n^{-2}\mathbb{E}|\mathrm{tr}\mathbb{E}_k\mathbf{A}_k^{-1}(z_1)\underline{\mathbf{A}}_k^{-1}(z_2) - \mathbb{E}\mathrm{tr}\mathbf{A}_k^{-1}(z_1)\underline{\mathbf{A}}_k^{-1}(z_2)|^2 \leq M/(n^2v^5),$$

$$(8.12) \quad n^{-2}\mathbb{E}|\mathrm{tr}\mathbb{E}_k\mathbf{A}_k^{-1}(z)\underline{\mathbf{A}}_k^{-2}(z) - \mathbb{E}\mathrm{tr}\mathbf{A}_k^{-1}(z)\underline{\mathbf{A}}_k^{-2}(z)|^2 \leq M/(n^2v^7),$$

$$(8.13) \quad n^{-2}\mathbb{E}|\mathrm{tr}\mathbf{A}_k^{-2}(z)\underline{\mathbf{A}}_k^{-2}(z) - \mathbb{E}\mathrm{tr}\mathbf{A}_k^{-2}(z)\underline{\mathbf{A}}_k^{-2}(z)|^2 \leq M/(n^2v^9).$$

and

$$(8.14) \quad |g(z)|^{-1}\mathbb{E}|\Gamma_1|^8 \leq M/(n^8v^{12}).$$

Remark 5. Checking on the argument of (8.11) shows that (8.11) is still true when the notation \mathbb{E}_k is removed.

Proof. We begin with a martingale decomposition of the random variable of interest:

$$\begin{aligned} & n^{-1}\mathrm{tr}\mathbf{A}_k^{-1}(z_1)\mathbb{E}_k\mathbf{A}_k^{-1}(z_2) - \mathbb{E}(n^{-1}\mathrm{tr}\mathbf{A}_k^{-1}(z_1)\mathbb{E}_k\mathbf{A}_k^{-1}(z_2)) \\ &= n^{-1}\sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) [\mathrm{tr}\mathbf{A}_k^{-1}(z_1)\mathbb{E}_k\mathbf{A}_k^{-1}(z_2)] \\ &= n^{-1}\sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) [\mathrm{tr}\mathbf{A}_k^{-1}(z_1)\mathbb{E}_k\mathbf{A}_k^{-1}(z_2) - \mathrm{tr}\mathbf{A}_{kj}^{-1}(z_1)\mathbb{E}_k\mathbf{A}_{kj}^{-1}(z_2)] \\ &= n^{-1}\sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(\delta_1 + \delta_2 + \delta_3), \end{aligned}$$

where, via (3.23),

$$\delta_1 = \beta_{kj}(z_1)\mathbf{s}_j^T\mathbf{A}_{kj}^{-1}(z_1)\mathbb{E}_k(\beta_{kj}(z_2)\mathbf{A}_{kj}^{-1}(z_2)\mathbf{s}_j\mathbf{s}_j^T\mathbf{A}_{kj}^{-1}(z_2))\mathbf{A}_{kj}^{-1}(z_1)\mathbf{s}_j$$

$$\delta_2 = -\beta_{kj}(z_1)\mathbf{s}_j^T\mathbf{A}_{kj}^{-1}(z_1)\mathbb{E}_k(\mathbf{A}_{kj}^{-1}(z_2))\mathbf{A}_{kj}^{-1}(z_1)\mathbf{s}_j$$

and

$$\delta_3 = -\mathrm{tr}\mathbf{A}_{kj}^{-1}(z_1)\mathbb{E}_k(\beta_{kj}(z_2)\mathbf{A}_{kj}^{-1}(z_2)\mathbf{s}_j\mathbf{s}_j^T\mathbf{A}_{kj}^{-1}(z_2)).$$

It follows from (3.25) that

$$(8.15) \quad |\delta_1| \leq v^{-2}.$$

This implies that when $j > k$,

$$(\mathbb{E}_j - \mathbb{E}_{j-1})\delta_1 = (\mathbb{E}_j - \mathbb{E}_{j-1})b_{12}(z_1)(\delta_{11} - \delta_{12}),$$

where $\delta_{12} = \xi_{kj}(z_1)\delta_1$ and

$$\delta_{11} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbb{E}_k(\beta_{kj}(z_2) \mathbf{G}_k(z_2)) \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbb{E}_k(\beta_{kj}(z_2) \mathbf{G}_k(z_2)) \mathbf{A}_{kj}^{-1}(z_1)$$

with $\mathbf{G}_k(z_2) = \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2)$. We conclude from (3.25), (8.15), (3.28) and Lemma 7 that

$$\mathbb{E} |n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(\delta_{11} + \delta_{12})|^2 \leq n^{-2} \sum_{j \neq k}^n \mathbb{E} |\delta_{11}|^2 + \mathbb{E} |\delta_{12}|^2 \leq M/(n^2 v^5).$$

For handling the case $j < k$, let

$$\alpha_{k1} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j, \quad \zeta_{kj1} = \alpha_{k1} - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2).$$

Applying (3.26) and the equality for $\underline{\beta}_{kj}(z_2)$ similar to (3.26) yields

$$\begin{aligned} (\mathbb{E}_j - \mathbb{E}_{j-1})\delta_1 &= (\mathbb{E}_j - \mathbb{E}_{j-1})[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \alpha_{k1}^2] \\ &= b_{12}(z_1) b_{12}(z_2) [\delta_{13} + 2\delta_{14} + \delta_{15} + \delta_{16} + \delta_{17}], \end{aligned}$$

where

$$\delta_{13} = (\mathbb{E}_j - \mathbb{E}_{j-1})(\zeta_{kj1}^2), \delta_{14} = (\mathbb{E}_j - \mathbb{E}_{j-1})(\zeta_{kj1} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \underline{\mathbf{A}}_{kj}^{-1}(z_2)),$$

$$\delta_{15} = -(\mathbb{E}_j - \mathbb{E}_{j-1})[\beta_{kj}(z_1) \xi_{kj}(z_1) \alpha_{k1}^2], \quad \delta_{16} = -(\mathbb{E}_j - \mathbb{E}_{j-1})[\underline{\beta}_{kj}(z_2) \underline{\xi}_{kj}(z_2) \alpha_{k1}^2]$$

and

$$\delta_{17} = (\mathbb{E}_j - \mathbb{E}_{j-1})[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) \underline{\xi}_{kj}(z_2) \alpha_{k1}^2].$$

It follows from Lemma 8 that

$$(8.16) \quad \mathbb{E} |\zeta_{kj1}|^4 \leq M/(n^2 v^6).$$

In view of (8.16) and (3.30),

$$\mathbb{E} |n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_{13})|^2 \leq M/(n^3 v^6).$$

While (3.41) and (3.30) yield

$$\mathbb{E} |n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_{14})|^2 \leq M/(n^2 v^5).$$

It follows from (3.25) that

$$|\beta_{kj}(z_1) \alpha_{k1} \alpha_{k2}| \leq M v^{-1} \|\underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j\|^2 = M v^{-1} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(\bar{z}_2) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j.$$

This, together with estimates similar to (3.41) and (8.16), ensures that

$$\mathbb{E}|n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_{15})|^2 \leq M/(n^2 v^5).$$

Obviously, this estimate applies to the term involving δ_{16} . From (3.25) and Lemma 8 we obtain

$$\mathbb{E}|n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_{17})|^2 \leq M/(n^3 v^6).$$

Summarizing the above we have

$$\mathbb{E}|n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_1)|^2 \leq M/(n^2 v^5).$$

Applying an argument similar to that for δ_1 in the case of $j > k$ one may prove that

$$\mathbb{E}|n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_2)|^2 \leq M/(n^2 v^5).$$

When $j > k$

$$n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_3) = 0.$$

When $j < k$, as in dealing with δ_1 in the case of $j > k$ one may verify that

$$\mathbb{E}|n^{-1} \sum_{j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_3)|^2 \leq M/(n^2 v^5).$$

Thus, the proof of (8.11) is complete.

As in (4.33), by Cauchy's theorem one may verify (8.12). Following the proof of (8.11) one can prove (8.13) and the details are omitted here.

Consider (8.14) next. Thanks to (3.24), it is enough to consider $\text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)$ rather than Γ_1 . As in (3.9) write

$$\begin{aligned} \text{tr} \mathbf{A}^{-1}(z) - \mathbb{E} \text{tr} \mathbf{A}^{-1}(z) &= - \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) (\beta_k \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k) \\ &= - \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) (b_1 \eta_k^{(2)}) + \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) (\beta_k b_1 \xi_k \mathbf{s}_k^T \mathbf{A}_k^{-2} \mathbf{s}_k), \end{aligned}$$

where the last step uses (3.5). It follows from (4.23), Lemmas 7, 8 and Burkholder's inequality that

$$n^{-8} |g(z)|^{-1} \mathbb{E} \left| \sum_{k=1}^n (\mathbb{E}_k - \mathbb{E}_{k-1}) (\eta_k^{(2)}) \right|^8 \leq n^{-12} v^{-8} |g(z)|^{-1} \mathbb{E} |M \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{A}_1^{-1}(\bar{z})|^4 \leq M/(n^8 v^{12}).$$

Similarly, via (3.7) we obtain

$$n^{-8}|g(z)|^{-1}\mathbb{E}\left|\sum_{k=1}^n(\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k \eta_k \mathbf{s}_k^T \mathbf{A}_k^{-2} \mathbf{s}_k)\right|^8 \leq M/(n^8 v^{12})$$

and via (3.4), (3.7) and Lemma 1

$$\begin{aligned} n^{-8}|g(z)|^{-1}\mathbb{E}\left|\sum_{k=1}^n(\mathbb{E}_k - \mathbb{E}_{k-1})(\beta_k \Gamma_k \mathbf{s}_k^T \mathbf{A}_k^{-2} \mathbf{s}_k)\right|^8 &\leq M n^{-4} v^{-8} |g(z)|^{-1} E |\Gamma_1|^8 \\ &\leq M n^{-4} v^{-8} |g(z)|^{-1} \mathbb{E} |n^{-1} \text{tr} \mathbf{A}^{-1}(z) - n^{-1} \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)|^8 + M/(n^8 v^{12}). \end{aligned}$$

Summarizing the above we have

$$(1 - M n^{-4} v^{-8}) \mathbb{E} |n^{-1} \text{tr} \mathbf{A}^{-1}(z) - n^{-1} \mathbb{E} \text{tr} \mathbf{A}^{-1}(z)|^8 \leq M/(n^8 v^{12}),$$

which implies (8.14). \square

9. APPENDIX 2

The aim in this section is to develop the asymptotic means and variances in Theorem 4 and Theorem 1. Consider (3.70) first. Note that

$$\begin{aligned} (3.70) = & -\frac{1}{2h^2\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'(\frac{x_1 - z_1}{h}) K'(\frac{x_2 - z_2}{h}) \\ & \times [\ln |\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)| + i \arg(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2))] dz_1 dz_2, \end{aligned}$$

where the contours \mathcal{C}_1 and \mathcal{C}_2 are two rectangles defined in (3.3) and (3.69), respectively.

As in Section 5 of [3] one may prove that

$$(9.1) \quad \inf_{z \in S, n} |\underline{m}_n^0(z)| > 0, \quad \left| \frac{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)}{z_1 - z_2} \right| \geq \frac{1}{2} |\underline{m}_n^0(z_1) \underline{m}_n^0(z_1)|,$$

where S is any bounded subset of \mathbb{C} .

To facilitate statements, denote the real parts of z_j by $u_j, j = 1, 2$. In what follows, let $n \rightarrow \infty$ first and then $v_0 \rightarrow 0$. Then, as argued in [3], the integrals in (9.1) involving the \arg term and the vertical sides approach zero.

Define

$$\begin{aligned} K_{ri}^{(1)} &= K'_r(\frac{x_1 - z_1}{h}) K'_r(\frac{x_2 - z_2}{h}) - K'_i(\frac{x_1 - z_1}{h}) K'_i(\frac{x_2 - z_2}{h}), \\ K_{ri}^{(2)} &= K'_r(\frac{x_1 - z_1}{h}) K'_r(\frac{x_2 - z_2}{h}) + K'_i(\frac{x_1 - z_1}{h}) K'_i(\frac{x_2 - z_2}{h}). \end{aligned}$$

Therefore it is enough to investigate the following integrals

$$-\frac{1}{h^2\pi^2} \int_{a_l}^{a_r} \int_{a_l - \varepsilon}^{a_r + \varepsilon} [K_{ri}^{(1)} \ln |\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)| - K_{ri}^{(2)} \ln |\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0(z_2)}|] du_1 du_2$$

$$(9.2) \quad = \frac{1}{h^2 \pi^2} \int_{a_l}^{a_r} \int_{a_l - \varepsilon}^{a_r + \varepsilon} (K'_r(\frac{x_1 - z_1}{h}) K'_r(\frac{x_2 - z_2}{h}) \ln \left| \frac{\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)}{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)} \right| du_1 du_2$$

$$(9.3) \quad + \frac{1}{h^2 \pi^2} \int_{a_l}^{a_r} \int_{a_l - \varepsilon}^{a_r + \varepsilon} (K'_i(\frac{x_1 - z_1}{h}) K'_i(\frac{x_2 - z_2}{h}) \times \ln \left| (\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2))(\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)) \right| du_1 du_2,$$

where $K'_r(h^{-1}(x - z))$ and $K'_i(h^{-1}(x - z))$, respectively, represent the real part and imaginary part of $K'(h^{-1}(x - z))$, $\overline{\underline{m}_n^0}(z)$ stands for the complex conjugate of $\underline{m}_n^0(z)$.

We develop the limit of (9.2) and (9.3) below. To this end, we list some facts below. By (2.6) and (2.7) one may verify that

$$(9.4) \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 \right| du_1 du_2 < \infty.$$

In addition, it follows from (2.6) that

$$(\ln h^{-2}) \int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K'_r(u_1) \int_{\frac{x-b-\varepsilon}{h}}^{\frac{x-a+\varepsilon}{h}} K'_r(u_2) du_1 du_2 \rightarrow 0.$$

This, together with (9.4), implies that as $n \rightarrow \infty$

$$(9.5) \quad \begin{aligned} & \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln(u_1 - u_2)^2 du_1 du_2 \\ &= \int_{\frac{x_1 - a_r}{h}}^{\frac{x_1 - a_l}{h}} \int_{\frac{x_2 - a_r - \varepsilon}{h}}^{\frac{x_2 - a_l + \varepsilon}{h}} K'(u_1) K'(u_2) \left[\ln(u_1 - u_2)^2 - \ln \frac{1}{h^2} \right] du_1 du_2 \\ &\rightarrow \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 du_1 du_2. \end{aligned}$$

By (2.9) and the continuity property of $K''(u + iv_0)$ and $K'(u + iv_0)$ in u and v_0 it is not difficult to prove that

$$(9.6) \quad \lim_{v_0 \rightarrow 0} \int_{-\infty}^{+\infty} |K''(u + iv_0)| du = \int_{-\infty}^{+\infty} |K''(u)| du$$

and

$$(9.7) \quad \lim_{v_0 \rightarrow 0} \int_{-\infty}^{+\infty} K^{(j)}(u + iv_0) du = \int_{-\infty}^{+\infty} K^{(j)}(u) du, \quad j = 0, 1,$$

where $K^{(j)}$ is the j -th derivative of K .

By complex Roller's theorem

$$(9.8) \quad K'_i(\frac{x - z_1}{h}) = K'_i(\frac{x - u_1}{h} + iv_0) = v K''_r(\frac{x - u}{h} + iv_1)$$

because $K'_i(h^{-1}(x-z)) = 0$, where v_1 lies in $(0, v_0)$. Thus we conclude from (9.1) and (9.6) that

$$\begin{aligned} & \left| \frac{1}{h} \int_{a_l}^{a_r} (K'_i(\frac{x_1 - z_1}{h}) \ln \left| (\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2))(\underline{m}_n^0(z_1) - \overline{\underline{m}}_n^0(z_2)) \right| du_1 \right| \\ & \leq v_0 h \ln(v_0^{-1}h) \frac{1}{h} \int_a^b |K''(\frac{x-u}{h} + iv_1)| du_1 \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, $v_0 \rightarrow 0$, which implies that (9.3) converges to zero.

Consider (9.2) next. As in (2.1) from (2.5) one may solve

$$\underline{m}(z) = \frac{-(z+1-c) + \sqrt{(z+1-c)^2 - 4z}}{2z}.$$

Note that the above equality still holds when c and $\underline{m}(z)$ are, respectively replaced by c_n and $\underline{m}_n^0(z)$. Also, when $z \rightarrow x \in [a, b]$ we have

$$(9.9) \quad \underline{m}(x) = \frac{-(x+1-c) + i\sqrt{(x-a)(b-x)}}{2x}.$$

It follows that for $u \in [(x-b)/h, (x-a)/h]$, as $n \rightarrow \infty$,

$$(9.10) \quad |\underline{m}_n^0(z_n) - \underline{m}(u_n)| \rightarrow 0,$$

where $z_n = u_n - iv_0 h$ with $u_n = x - uh$.

Now, as in [3], for (9.2) write

$$(9.11) \quad \ln \left| \frac{\underline{m}_n^0(z_1) - \overline{\underline{m}}_n^0(z_2)}{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)} \right| = \frac{1}{2} \ln \left(1 + \frac{4\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)|^2} \right),$$

where $\underline{m}_{ni}^0(z)$ denotes the imaginary part of $\underline{m}_n^0(z)$. By (9.1)

$$(9.12) \quad \ln \left(1 + \frac{4\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)|^2} \right) \leq \ln \left(1 + \frac{16\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{(u_1 - u_2)^2 |\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)|^2} \right).$$

In view of (9.1) and Lemma 8

$$(9.13) \quad \sup_{u_1, u_2 \in [a, b], v_1, v_2 \in [v_0 h, 1]} \left| \frac{\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)|^2} \right| < \infty.$$

By the generalized dominated convergence theorem we then conclude from (9.5), (9.7), (9.10), (9.12), (9.13) that as $n \rightarrow \infty$

$$\begin{aligned} & \int_{\frac{x_1 - a_r}{h}}^{\frac{x_1 - a_l}{h}} \int_{\frac{x_2 - a_r - \varepsilon}{h}}^{\frac{x_2 - a_l + \varepsilon}{h}} K'_r(z_1) K'_r(z_2) \left[\ln \left| \frac{\underline{m}_n^0(u_{n1} - iv_0 h) - \overline{\underline{m}}_n^0(u_{n2} - iv_0 h/2)}{\underline{m}_n^0(u_{n1} - iv_0 h) - \underline{m}_n^0(u_{n2} - iv_0 h/2)} \right| \right. \\ & \quad \left. - \ln \left| \frac{\underline{m}(u_{n1}) - \overline{\underline{m}}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| \right] du_1 du_2 \rightarrow 0, \end{aligned}$$

where $u_{nj} = x_j - u_j h$, $j = 1, 2$. In addition, it follows from (9.5), (9.7), and inequalities similar to (9.12) and (9.13) that as $n \rightarrow \infty$ and then $v_0 \rightarrow 0$

$$\int_{\frac{x_1 - a_r}{h}}^{\frac{x_1 - a_l}{h}} \int_{\frac{x_2 - a_r - \varepsilon}{h}}^{\frac{x_2 - a_l + \varepsilon}{h}} (K'_r(z_1) K'_r(z_2) - K'_r(u_1) K'_r(u_2)) \ln \left| \frac{\underline{m}(u_{n1}) - \overline{\underline{m}}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| du_1 du_2 \rightarrow 0.$$

Therefore (9.2) can be reduced to the following

$$(9.14) \quad \int_{\frac{x_1-a_r}{h}}^{\frac{x_1-a_l}{h}} \int_{\frac{x_2-a_r-\varepsilon}{h}}^{\frac{x_2-a_l+\varepsilon}{h}} K'(u_1)K'(u_2) \ln \left| \frac{\underline{m}(u_{n1}) - \overline{m}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| du_1 du_2 + o(1),$$

which turns to be

$$\frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'(\frac{u_1}{h})K'(\frac{u_2}{h}) \ln \left| \frac{\underline{m}(x_1-u_1) - \overline{m}(x_2-u_2)}{\underline{m}(x_1-u_1) - \underline{m}(x_2-u_2)} \right| du_1 du_2 + o(1).$$

To handle (9.14), we need one more lemma:

Lemma 10. *Suppose that the function $g(x_1, x_2)$ is continuous in x_1 and x_2 ,*

$$(9.15) \quad \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r}^{x_2-a_l} |g(x_1-u_1, x_2-u_2)| du_1 du_2 < \infty$$

and

$$(9.16) \quad \int_{x_1-a_r}^{x_1-a_l} |g(x_1-u_1, x_2)| du_1 < \infty, \quad \int_{x_2-a_r}^{x_2-a_l} |g(x_1, x_2-u_2)| du_2 < \infty.$$

Then, as $n \rightarrow \infty$

$$(9.17) \quad \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'(\frac{u_1}{h})K'(\frac{u_2}{h})g(x_1-u_1, x_2-u_2) du_1 du_2 \rightarrow 0,$$

where $x_1 \neq a_l, a_r$ and $x_2 \neq a_l, a_r$.

Proof. Define the sets $G_1 = (|u_1| \leq \delta_1) \cap (|u_2| > \delta_2)$, $G_2 = (|u_1| > \delta_1) \cap (|u_2| \leq \delta_2)$ and $G_3 = (|u_1| > \delta_1) \cap (|u_2| > \delta_2)$. Splitting the region of integration into the union of the sets $(|u_1| \leq \delta_1) \cap (|u_2| \leq \delta_2)$, G_1 , G_2 and G_3 gives

$$(9.18) \quad \left| \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'(\frac{u_1}{h})K'(\frac{u_2}{h}) \left[g(x_1-u_1, x_2-u_2) - g(x_1, x_2) \right] du_1 du_2 \right| \leq I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= \sup_{|u_1| \leq \delta_1, |u_2| \leq \delta_2} \left| g(x_1-u_1, x_2-u_2) - g(x_1, x_2) \right| \int_{-\infty}^{+\infty} |K'(u)| du \Big|^2, \\ I_2 &= |g(x_1, x_2)| \left| \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} I(G_1 \cup G_2 \cup G_3) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) du_1 du_2 \right|, \\ I_3 &= \left| \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} I(G_1) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) g(x_1-u_1, x_2-u_2) du_1 du_2 \right|, \\ I_4 &= \left| \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} I(G_2) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) g(x_1-u_1, x_2-u_2) du_1 du_2 \right| \end{aligned}$$

and

$$I_5 = \left| \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} I(G_3) \frac{u_1 u_2}{h^2} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \frac{g(x_1-u_1, x_2-u_2)}{u_1 u_2} du_1 du_2 \right|.$$

Evidently, $I_1 \rightarrow 0$ due to the continuity property of $g(x_1, x_2)$ when δ_1 and δ_2 converge to zero. As $n \rightarrow \infty$, for I_2 we have

$$I_2 \leq M|g(x_1, x_2)| \int_{|u|>\delta/h} |K'(u)|du \int_{-\infty}^{+\infty} |K'(u)|du \rightarrow 0,$$

and for I_5 by (9.16) we obtain

$$\begin{aligned} I_5 &\leq (\delta_1 \delta_2)^{-1} \sup_{|u_1|>\delta_1/h} |u_1 K'(u_1)| \sup_{|u_2|>\delta_2/h} |u_2 K'(u_2)| \\ &\quad \times \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} |g(x_1 - u_1, x_2 - u_2)| du_1 du_2 \rightarrow 0. \end{aligned}$$

Consider I_3 . Similar to I_5 ,

$$I_3 \leq \delta_2^{-1} \sup_{|u_2|>\delta_2/h} |u_2 K'(u_2)| \int_{|u_1|\leq\delta_1/h} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} |K'(u_1)g(x_1 - u_1 h, x_2 - u_2)| du_1 du_2.$$

While, as $n \rightarrow \infty$ and then $\delta_1 \rightarrow 0$, by the dominated convergence theorem

$$h^{-1} \int_{|u_1|\leq\delta_1} |K'(\frac{u_1}{h})| \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} |(g(x_1 - u_1, x_2 - u_2) - g(x_1, x_2 - u_2))| du_1 du_2 \rightarrow 0.$$

From (9.16) we then see that $I_3 \rightarrow 0$. One may similarly prove that I_4 converges to zero as well. We summarize the above that (9.18) converges to zero as $n \rightarrow \infty$ first and then both $\delta_1 \rightarrow 0$ and $\delta_2 \rightarrow 0$. In addition, apparently,

$$\begin{aligned} (9.19) \quad g(x_1, x_2) h^{-1} \int_{x_1-a_r}^{x_1-a_l} K'(\frac{u}{h}) du &= g(x_1, x_2) \int_{\frac{x_1-a_l}{h}}^{\frac{x_1-a_r}{h}} K'(u) du \\ &= g(x_1, x_2) K(u) \Big|_{\frac{x_1-a_l}{h}}^{\frac{x_1-a_r}{h}} \rightarrow 0. \end{aligned}$$

Thus (9.17) is proved. \square

We are now in a position to apply Lemma 10 to (9.14). It follows from (9.9) that $\underline{m}(x_1) \neq \underline{m}(x_2)$ and $\underline{m}(x_1) \neq \overline{m}(x_2)$ whenever $x_1 \neq x_2$. Also, note (5.1) in [3]. Therefore $g(x_1, x_2) = \ln(|\underline{m}(x_1) - \overline{m}(x_2)||\underline{m}(x_1) - \underline{m}(x_2)|^{-1})$ is continuous in x_1 and x_2 . Furthermore, it is straightforward to show that $\ln(1 + M((x_1 - x_2) - (u_1 - u_2))^{-1})$ for $u_1, u_2 \in [a_l - \varepsilon, a_r + \varepsilon]$ is Lebesgue integrable and $\ln(1 + M((x_1 - x_2) - u_1)^{-1})$ for $u_2 \in [a_l - \varepsilon, a_r + \varepsilon]$ is Lebesgue integrable. Thus, in view of inequalities similar to (9.11)-(9.13) and applying (9.17) we have

$$(9.20) \quad \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 \rightarrow 0,$$

which is the limit of (9.2) due to (9.14) when $x_1 \neq x_2$.

When $x_1 = x_2 = x$ taking $g(x_1, x_2) = \ln|\underline{m}(x) - \overline{m}(x)|$ and applying (9.17) we obtain

$$(9.21) \quad \frac{1}{h^2} \int_{x-a_r}^{x-a_l} \int_{x-a_r-\varepsilon}^{x-a_l+\varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln |\underline{m}(x - u_1) - \overline{m}(x - u_2)| du_1 du_2 \rightarrow 0.$$

Here we keep in mind that the boundary points are not considered when investigating the case $x_1 = x_2 = x$. Consider next

$$(9.22) \quad \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) \ln |\underline{m}(x-u_1) - \underline{m}(x-u_2)| du_1 du_2.$$

By complex Roller's theorem we have

$$(9.23) \quad \begin{aligned} & \ln |\underline{m}(x-u_1) - \underline{m}(x-u_2)| \\ &= 2^{-1} \ln ((u_1-u_2)^2 [|\underline{m}'_r(x-u_3)|^2 + |\underline{m}'_i(x-u_4)|^2]) \\ &= 2^{-1} \ln(u_1-u_2)^2 + 2^{-1} g_{ri}(x-u_1, x-u_2), \end{aligned}$$

where $g_r(x-u_1, x-u_2) = \ln (|\underline{m}'_r(t_1(x-u_1) + (1-t_1)(x-u_2))|^2 + |\underline{m}'_i(t_2(x-u_1) + (1-t_2)(x-u_2))|^2)$, $u_3 = t_1 u_1 + (1-t_1)u_2$, $u_4 = t_2 u_1 + (1-t_2)u_2$ and $t_1, t_2 \in (0, 1)$. It follows from inequalities for $\underline{m}(x)$ similar to (9.1) that

$$\left| \int_{x-b}^{x-a} \int_{x-b-\varepsilon}^{x-a+\varepsilon} \ln |\underline{m}(x-u_1) - \underline{m}(x-u_2)| du_1 du_2 \right| < \infty.$$

This, together with (9.23), ensures that

$$\left| \int_{x-b}^{x-a} \int_{x-b-\varepsilon}^{x-a+\varepsilon} g_r(x-u_1, x-u_2) du_1 du_2 \right| < \infty.$$

Similarly, one may check the remaining conditions in Lemma 10. Therefore, using Lemma 10 with $g(x_1, x_2) = \ln |\underline{m}'(x)|^2$ gives

$$(9.24) \quad \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) g_r(x-u_1, x-u_2) du_1 du_2 \rightarrow 0.$$

We then conclude from (9.23), (9.24) and (9.5) that

$$(9.22) = \frac{1}{2} \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) \ln(u_1-u_2)^2 du_1 du_2 + o(1)$$

$$(9.25) \quad \rightarrow \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1) K'(u_2) \ln(u_1-u_2)^2 du_1 du_2.$$

which is minus the limit of (9.2) due to (9.21) and (9.14) when $x_1 = x_2$.

Limit of (5.15). From an expression similar to (2.5) we obtain

$$\frac{d}{dz} \underline{m}_n^0(z) = (\underline{m}_n^0(z))^2 \left(1 - c \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} \right)^{-1}.$$

It follows that (5.15) becomes

$$(9.26) \quad \begin{aligned} & \frac{1}{4\pi i} \oint K(\frac{x-z}{h}) \frac{d}{dz} \ln \left[1 - c \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} \right] dz \\ &= \frac{1}{4\pi h i} \oint K'(\frac{x-z}{h}) \ln \left[1 - c \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} \right] dz. \end{aligned}$$

In view of (4.12) and (4.4) we see that the integrals on the two vertical lines in (9.26) are bounded by $Mv \ln v^{-1}$, which converges to zero as $v \rightarrow 0$. The integrals on the two horizontal lines are equal to

$$(9.27) \quad \frac{1}{2\pi h} \int K'_i\left(\frac{x-z}{h}\right) \ln \left| 1 - c \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} \right| du$$

$$(9.28) \quad + \frac{1}{2\pi h} \int K'_r\left(\frac{x-z}{h}\right) \arg \left[1 - c \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} \right] du.$$

By (4.12), (4.4) and (9.8), (9.27) is bounded by $Mv \ln v^{-1}$, converging to zero. It follows from (9.10) that

$$\frac{(\underline{m}_n^0(z_n))^2}{(1 + \underline{m}_n^0(z_n))^2} - \frac{(\underline{m}(u_n))^2}{(1 + \underline{m}(u_n))^2} \rightarrow 0.$$

We then conclude from the dominated convergence theorem that

$$\int K'_r(z) \left[\arg \left(1 - c \frac{(\underline{m}_n^0(z_n))^2}{(1 + \underline{m}_n^0(z_n))^2} \right) - \arg \left(1 - c \frac{(\underline{m}(u_n))^2}{(1 + \underline{m}(u_n))^2} \right) \right] du \rightarrow 0.$$

Moreover, by (9.7) we obtain

$$\int (K'_r(z) - K'_r(u)) \arg \left[1 - c \frac{(\underline{m}(u_n))^2}{(1 + \underline{m}(u_n))^2} \right] du \rightarrow 0.$$

By (9.19) and Theorem 1A in [20] (replacing $K(x)$ there by $K'(x)$) we see that

$$\int K'_r(u) \arg \left[1 - c \frac{(\underline{m}(u_n))^2}{(1 + \underline{m}(u_n))^2} \right] du \rightarrow 0.$$

Summarizing the above yields that (5.15) converges to zero.

Limits of (6.6) and (6.8). Repeating the argument leading to (9.14) yields that (6.6) becomes

$$(9.29) \quad \frac{1}{h^2 \ln h^{-1}} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 + o(1).$$

The argument of (9.18) in Lemma 10 indeed also, together with (2.7), gives

$$(9.30) \quad \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) g(x_1 - u_1, x_2 - u_2) du_1 du_2 - g(x_1, x_2) \rightarrow 0.$$

This ensures that (9.29) converges to zero when $x_1 \neq x_2$. When $x_1 = x_2 = x$, by (9.30) we have

$$\frac{1}{h^2 \ln h^{-1}} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2) \right| du_1 du_2 \rightarrow 0.$$

Applying (9.30) and replacing $K'(x)$ in (9.5), (9.23), (9.24) and (9.25) by $K(x)$, we can prove that

$$-\frac{1}{h^2 \ln h^{-1}} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2) \right| du_1 du_2 \rightarrow 1.$$

Moreover, from the conditions on h one may show that

$$\frac{\ln n}{\ln h^{-1}} \rightarrow 2.$$

Checking on the argument of (5.15) and replacing $K'(x)$ there with $K(x)$, along with (6.5), we have $(6.8) \rightarrow 0$. Thus the proof is complete.

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